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ON THE UTILITY OF [0,1]-VALUED CONTINUOUS LIFETIME DISTRIBUTIONS: A REVIEW

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ABSTRACT

Recent studies on the theory of statistical distribution reveal a wide application of continuous lifetime distributions with support [0,1] in real world data fittings. Bounded lifetime distributions such as Kumaraswamy distribution, Beta distribution, one-parameter Topp-Leone distribution and the recently developed continuous Bernoulli distribution have gained popularity in modelling real datasets taking the form of proportions, percentages, probabilities, etc.
Several attempts have been made to generalize these continuous lifetime distributions in order to increase their chance of providing good fit in real life data analysis. This paper presents a general review on the utility of [0,1]-valued lifetime distributions.

1. Introduction

Lifetime distributions amongst the numerous lists of statistical models have gained a considerably high attention in the theory of statistical analysis of real datasets. In practice, it has been applied in variety of fields such as Demography, Biological Sciences, Engineering, Actuarial Sciences, Economics, etc. Examples of these lifetime distributions include the Weibull, Gumbel, Beta, Gamma, Lindley, Logistic, Kumaraswamy, Burr XII distributions, etc. Each of these models play a significant role in data fitting based on their uniqueness and statistical properties. In real world phenomena, data sets often take values either within the interval $[0,1]$, $[0,\infty)$ or $(-\infty,\infty)$. In clear terms, the unit interval $[0,1]$ are referred to as proportional or percentage data while the interval $(-\infty,\infty)$ are generalized as the extreme value data.

Whichever case, these models have been studied and applied to model real world situations. Undoubtedly, in many scenarios, they have proved their usefulness in real life data fittings. However, there are rigid cases where these classical lifetime distributions will fail to provide good fits. Thus, the advent of several methods ofgeneralizing classical distributions to increase their flexibility in real life data fitting. Evidently, it is established that the flexibility of classical distributions is increased by introducing extra parameter(s) to the

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distribution. This paper focuses on the generalization of lifetime distributions defined on a unit interval. In what follows, a review of bounded distributions such as the Beta distribution, Kumaraswamy distribution, one-parameter Topp-Leone distribution and continuous Bernoulli distribution is presented.

2. Review of the Literature

2.1 The Beta Distribution

The Beta distribution is one of the univariate continuous distributions used in the analysis of data sets defined on the unit interval $[0,1]$. The density function of the Beta distribution of the first kind has been defined by [1] as:

$$f(x)=\frac{1}{B(\theta,\lambda)}x^{\theta-1}(1-x)^{\lambda-1}, \qquad 0<x<1, \quad \theta,\lambda>0, \tag{1}$$

where $B(\theta,\lambda)$ is the beta function obtained by

$$B(\theta,\lambda)=\int_0^1 x^{\theta-1}(1-x)^{\lambda-1}\,dx.$$

The use of beta distribution as a prior distribution for binomial proportions in Bayesian analysis. has been suggested by [2]. The model was utilized by [3] to calculate the expected cost from a civil or industrial engineering project, The beta distribution has been used in the study of population genomics by [4]. The application of the model for input process within a stochastic simulation was presented in [5].

The beta distribution was considered as a generator to propose the beta-G family of distributions in [6]. Suppose that the random variable X in equation (1) is a baseline distribution $G(x)$, the cumulative distribution function (cdf) of the beta-G family of distributions was defined by [6] as:

$$G(x)=\frac{1}{B(\theta,\lambda)}\int_0^{F(x)} x^{\theta-1}(1-x)^{\lambda-1}\,dx, \tag{2}$$

and the probability density function(pdf) as:

$$g(x)=\frac{f(x)}{B(\theta,\lambda)}\left[F(x)\right]^{\theta-1}\left[1-F(x)\right]^{\lambda-1}, \qquad x>0, \quad \theta,\lambda>0. \tag{3}$$

Several authors have leveraged on the technique defined in equations (2) and (3) to develop an extension of the beta distribution. Table 1 gives a compilation of some of these generalized distributions.

Table 1: Generalized Lifetime Distributions using the Beta-*G* Framework

Distributions	Probability density function(pdf)	Authors
Beta Normal Distribution (BND)	$\dfrac{\phi\left(\frac{x-\mu}{\sigma}\right)}{B(a,b)}\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{a-1}\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{b-1}$	[6]
Beta Gumbel Distribution (BGD)	$\dfrac{\lambda}{\theta B(a,b)}\exp(-a\lambda)\left[1-\exp(-\lambda)\right]^{b-1}$, $\lambda=\exp(-(x-\mu)/\theta)$	[7]
Beta Weibull Distribution (BWD)	$\dfrac{e^{-(x/\gamma)^c}}{B(a,b)}\dfrac{c}{\gamma}\left(\dfrac{x}{\gamma}\right)^{c-1}\left[1-e^{-(x/\gamma)^c}\right]^{a-1}\left[e^{-(x/\gamma)^c}\right]^{b-1}$	[8]
Beta Exponential Distribution (BED)	$\dfrac{\lambda}{B(a,b)}\exp(-a\lambda x)\left[1-\exp(-\lambda x)\right]^{b-1}$	[9]

Beta Pareto Distribution (BPrD)	$\frac{k}{\gamma B(a,b)}\left[1-\left(\frac{x}{\gamma}\right)^{-k}\right]^{a-1}\left(\frac{x}{\gamma}\right)^{-kb-1}$	[10]
Beta Modified Weibull Distribution (BMWD)	$\frac{\alpha x^{\gamma-1}(\gamma+\lambda x)e^{\lambda x}}{B(a,b)}\left[1-\exp\left(-\alpha x^{\gamma}e^{\lambda x}\right)\right]^{a-1} \times \exp\left(-b\alpha x^{\gamma}e^{\lambda x}\right)$	[11]
Beta Weibull Geometric Distribution (BWGD)	$\frac{c(1-p)^{b}\gamma^{c}x^{c-1}e^{-b(\gamma x)^{c}}\left(1-e^{-(\gamma x)^{c}}\right)^{b-1}}{B(a,b)\left(1-pe^{-(\gamma x)^{c}}\right)^{a+b}}$	[12]
Beta Power Distribution (BPwD)	$\frac{\theta\lambda}{B(a,b)}(\lambda x)^{\theta a-1}\left[1-(\lambda x)^{\theta}\right]^{b-1}$	[13]
Beta Power Exponential Distribution (BPED)	$\frac{\lambda}{B(a,b+1)}\left[1-\exp(-\lambda x)\right]^{a-1}\left[\exp(-\lambda x)\right]^{b+1}$	[14]
Beta Exponentiated Lindley Distribution (BELD)	$\frac{\theta^{2}\lambda(1+x)e^{-\theta x}}{B(a,b)(\theta+1)}\left[1-\left(\frac{1+\theta+\theta x}{\theta+1}\right)e^{-\theta x}\right]^{\lambda a-1} \times\left[\left(\frac{1+\theta+\theta x}{\theta+1}\right)e^{-\theta x}\right]^{\lambda b-1}$	[15]
Beta Power Lindley Distribution (BPLD)	$\frac{\alpha\theta^{2}\left(1+x^{\alpha}\right)x^{\alpha-1}e^{-\theta x^{\alpha}}}{B(a,b)(\theta+1)}\left[1-\left(\frac{1+\theta+\theta x}{\theta+1}\right)e^{-\theta x}\right]^{a-1} \times\left[\left(\frac{1+\theta+\theta x}{\theta+1}\right)e^{-\theta x}\right]^{b-1}$	[16]
Beta Lindley Distribution (BLD)	$\frac{\theta^{2}(1+x)e^{-\theta x}}{B(a,b)(\theta+1)}\left[1-\left(\frac{1+\theta+\theta x}{\theta+1}\right)e^{-\theta x}\right]^{a-1} \times\left[\left(\frac{1+\theta+\theta x}{\theta+1}\right)e^{-\theta x}\right]^{b-1}$	[17]
Beta Sarhan-Zaindln Modified Weibull Distribution (BSZMWD)	$\frac{\left(\lambda+\beta kx^{k-1}\right)}{B(a,b)}\exp\left(-b\lambda x-b\beta x^{k}\right) \times\left[1-\exp\left(-\lambda x-\beta x^{k}\right)\right]^{a-1}$	[18]
Beta Exponential Frechet Distribution (BEFD)	$\frac{\alpha\theta^{\alpha}\lambda x^{-(\alpha+1)}e^{-\left(\frac{\theta}{x}\right)^{\alpha}}}{B(a,b)}\left[1-e^{-\left(\frac{\theta}{x}\right)^{\alpha}}\right]^{\lambda b-1} \times\left[1-\left[1-e^{-\left(\frac{\theta}{x}\right)^{\alpha}}\right]^{\lambda}\right]^{a-1}$	[19]

Beta Generalized ExponentiatedFrechet Distribution (BGEFD)	$\frac{\alpha\theta\lambda\beta x^{-(\alpha+1)}e^{-x^{-\alpha}}}{B(a,b)}\left[1-e^{-x^{-\alpha}}\right]^{\lambda\theta-1} \times\left[1-\left[1-e^{-x^{-\alpha}}\right]^{\lambda\theta}\right]^{\beta a-1} \times\left[1-\left[1-\left[1-e^{-x^{-\alpha}}\right]^{\lambda\theta}\right]^{\beta}\right]^{b-1}$	[20]
Beta Exponentiated Modified Weibull Distribution (BEMWD)	$\frac{\gamma\left(\delta+\lambda\theta^{\lambda}x^{\lambda-1}\right)e^{-\left(\delta x+(\theta x)^{\lambda}\right)}}{B(a,b)}\left[1-e^{-\left(\delta x+(\theta x)^{\lambda}\right)}\right]^{a\gamma-1} \times\left[1-\left[1-e^{-\left(\delta x+(\theta x)^{\lambda}\right)}\right]^{\gamma-1}\right]^{b-1}$	[21]

A study conducted on the shapes of the density function of the classical beta distribution revealed that the distribution accommodates a left-skewed, right-skewed and symmetric unimodal shapes, [22]. The list of beta-generalized distributions presented in Table 1 have been reported to have same shapes for their density functions. In particular, the density plots of the beta-normal distribution is said to exhibit a bimodal property. A major drawback of the classical beta distribution and its generalizations is that the cumulative distribution function does not exist in close form and thus, not invertible.

2.2 The Kumaraswamy Distribution

A two-parameter continuous lifetime distribution called the Kumaraswamy distribution defined on a unit interval [0,1] was developed by [23]. Similar to the beta distribution, the distribution is characterized with two shape parameters with the cumulative distribution function (cdf) and probability density function (pdf), respectively defined as

$$F(x,\theta,\lambda)=1-\left(1-x^{\theta}\right)^{\lambda}, \qquad \theta,\lambda>0,\ 0<x<1, \tag{4}$$

and

$$f(x,\theta,\lambda) = \lambda\theta x^{\theta-1}\left(1-x^{\theta}\right)^{\lambda-1}, \qquad \theta,\lambda>0,\ 0<x<1. \tag{5}$$

A detailed study of theKumaraswamy distribution and an outline of its advantageous features over the beta distribution are provided in [24]. Contrary to the beta distribution, the Kumaraswamy distribution has a closed form expression for the cdf which is equally invertible. Motivated by the work of [6], a family of distributions was introduced by considering the Kumaraswamy distribution as a generator, [12]. The cdf and pdfof the Kumaraswamy-G (Kum-G) family of distributions was defined respectively as:

$$F(x,\theta,\lambda,\xi)=1-\left(1-G(x,\xi)^{\theta}\right)^{\lambda}, \qquad \theta,\lambda>0,\ x>0. \tag{6}$$

and

$$f(x,\theta,\lambda,\xi) = \lambda\theta g(x,\xi)\left[G(x,\xi)\right]^{\theta-1}\left(1-\left[G(x,\xi)\right]^{\theta}\right)^{\lambda-1}, \qquad \theta,\lambda>0,\ x>0. \tag{7}$$

The Kum-G framework has been widely explored to generalize existing classical lifetime distributions. Table 2 shows some of the generalized distributions using the Kum-*G* framework.

Table 2: Generalized Lifetime Distributions using the Kum-*G* Framework

Distributions	Cumulative distribution function (cdf)	Authors
KumaraswamyWeibull Distribution (KWD)	$1-\left\{1-\left(1-e^{-(\theta x)^{\beta}}\right)^{\alpha}\right\}^{\lambda}$	[25]
Kumaraswamy Generalized Gamma Distribution (KGGD)	$1-\left\{1-\left(\gamma\left[k,\left(\frac{x}{\beta}\right)^{\theta}\right]\right)^{\alpha}\right\}^{\lambda}$, $\gamma[k,x]=\int_0^x u^{k-1}e^{-u}du$	[26]
KumaraswamyGumbel Distribution (KGD)	$1-\{1-\exp(-\alpha u)\}^{\lambda}$, $u=\exp\left(\frac{x-\mu}{\sigma}\right)$	[27]
Kumaraswamy Inverse Weibull Distribution (KIWD)	$1-\left\{1-e^{-\alpha\left(\frac{\theta}{x^{\beta}}\right)}\right\}^{\lambda}$	[28]
Kumaraswamy Burr XII Distribution (KBXIID)	$1-\left\{1-\left(1-\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-k}\right)^{\alpha}\right\}^{\lambda}$	[29]
Kumaraswamy Quasi Lindley Distribution (KQLD)	$1-\left\{1-\left(1-\frac{(1+\lambda+\theta x)}{\lambda+1}e^{-\theta x}\right)^{\alpha}\right\}^{\lambda}$	[30]
Kumaraswamy-Pareto Distribution (KKD)	$1-\left\{1-\left(1-\left(\frac{\beta}{x}\right)^{k}\right)^{\alpha}\right\}^{\lambda}$	[31]
Kumaraswamy-Kumaraswamy Distribution (KKD)	$1-\left\{1-\left(1-\left(1-x^{\beta}\right)^{\theta}\right)^{\alpha}\right\}^{\lambda}$	[32]
Kumaraswamy Lindley Poisson Distribution (KLPD)	$1-\left\{1-\left(\frac{1-\exp\left(\lambda\left(1-\frac{(1+\theta+\theta x)}{\theta+1}e^{-\theta x}\right)\right)}{1-e^{\lambda}}\right)^{\alpha}\right\}^{\lambda}$	[33]
Kumaraswamy Power Lindley Distribution (KPLD)	$1-\left\{1-\left(1-\frac{(1+\theta+\theta x^{\lambda})}{\theta+1}e^{-\theta x\lambda}\right)^{\alpha}\right\}^{\lambda}$	[34]
Kumaraswamy Burr III Distribution (KBIIID)	$1-\left\{1-\left[1+(x)^{-c}\right]^{-k\alpha}\right\}^{\lambda}$	[35]
Kumaraswamy Lindley Distribution (KLD)	$1-\left\{1-\left(1-\frac{(1+\theta+\theta x)}{\theta+1}e^{-\theta x}\right)^{\alpha}\right\}^{\lambda}$	[36]
Kumaraswamy Unit-Gompertz Distribution (KUGD)	$1-\left[1-e^{-\gamma(x^{-\beta}-1)}\right]^{\lambda}$	[37]

The Kumaraswamy-generated lifetime distributions listed in Table 2 have been applied to model Engineering, Actuarial Sciences, Economics, health, etc, related data sets. Graphical representations of the density functions of these distributions have also been found to accommodate a left-skewed, right-skewed, symmetric shapes, while the plots of the hazard rate function reveal an increasing, decreasing, bathtub and inverted bathtub-shaped property.

2.3 TheTopp-Leone Distribution

A two-parameter reversed-J continuous lifetime distribution was developed by [38] with the cdf and pdf defined respectively as:

$$F(x,a,b)=\left(\frac{x}{a}\right)^{b}\left(2-\frac{x}{a}\right)^{b}, \qquad 0\le x\le a<\infty,\ b>0, \tag{8}$$

and

$$f(x,a,b)=\frac{2b}{a}\left(\frac{x}{a}\right)^{b-1}\left(1-\frac{x}{a}\right)\left(2-\frac{x}{a}\right)^{b-1}, \qquad 0\le x\le a<\infty,\ b>0, \tag{9}$$

By fixing $a=1$, the one-parameter Topp-Leone distribution with bounded support is defined by the cdf and pdf respectively as

$$F(x,b)=(x)^{b}(2-x)^{b}, \qquad 0\le x\le 1,\ b>0, \tag{10}$$

and

$$f(x,b)=2b(x)^{b-1}(1-x)(2-x)^{b-1}, \qquad 0\le x\le 1,\ b>0. \tag{11}$$

The one-parameter Topp-Leone distribution happens to be the simplest distribution with a bathtub hazard rate property and this unique feature has attracted the attention of many researchers. Employing the idea used in [6] and [12]. The Topp-Leone-G family of distributions has been studied in [39] with its cdf defined by

$$F(x,b,\xi)=\left[G(x,\xi)\right]^{b}\left[2-G(x,\xi)\right]^{b}, \qquad \xi,b>0,\ x>0, \tag{12}$$

and the corresponding pdf obtained as

$$f(x,b,\xi)=2bg(x,\xi)\left[1-G(x,\xi)\right]\left[G(x,\xi)\right]^{b-1}\left[2-G(x,\xi)\right]^{b-1}, \qquad \xi,b>0,\ x>0. \tag{13}$$

Table 3 presents a collection of generalized distributions developed using the Topp-Leone-G framework defined in equations (12) and (13).

Table 3: Generalized Lifetime Distributions using the Topp-Leone-*G* Framework

Distributions	**Cumulative distribution function (cdf)**	**Authors**
Topp-Leone Gen. Exp. Distribution (TLGED)	$\left[1-e^{-\lambda x}\right]^{\beta b}\left[2-\left(1-e^{-\lambda x}\right)^{\beta}\right]^{b}$	[39]
Topp-Leone Exp. Distribution (TLED)	$[1-e^{-2\lambda x}]^{b}$	[40]
Topp-Leone Gumbel Distribution (TLGD)	$exp\left\{-b\ exp\left(-\frac{x-\mu}{\sigma}\right)\right\} \times\left(2-exp\left\{-exp\left(-\frac{x-\mu}{\sigma}\right)\right\}\right)^{b}$	[41]
Topp-Leone Generated Weibull Distribution (TLGWD)	$\left(1-e^{-(\lambda x)^{\beta}}\right)^{b\theta} \times\left[2-\left(1-e^{-(\lambda x)^{\beta}}\right)^{\theta}\right]^{b}$	[42]
Topp-Leone Bur XII Distribution	$\left[1-(1+x^{\beta})^{-2\lambda}\right]^{b}$	

(TLBXIID)		[43]
Topp-Leone Nadajarah-Haghighi Distribution (TLNHD)	$\left[1-exp\{2(1-\ (1+\beta x)^{\lambda})\}\right]^{b}$	[44]
Topp-Leone Inverse Weibull Distribution (TLIWD)	$\left(1-\left[1-exp\left\{-\frac{\beta}{x^{\lambda}}\right\}\right]^{2}\right)^{b}$	[45]
Topp-Leone Exp. Power Lindley Distribution (TLEPLD)	$\left(1-\frac{1+\beta+\beta x^{\lambda}}{1+\beta}e^{-\beta x^{\lambda}}\right)^{\theta b}\times \left[2-\left(1-\frac{1+\beta+\beta x^{\lambda}}{1+\beta}e^{-\beta x^{\lambda}}\right)^{\theta}\right]^{b}$	[46]
Topp-Leone Normal Distribution (TLND)	$\left\{\phi\left(\frac{x-\mu}{\sigma}\right)\left[2-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]\right\}^{b}$	[47]
Topp-Leone Linear Exponential Distribution (TLLED)	$\left(1-e^{-2\left(\theta x+\frac{\beta x^{2}}{2}\right)}\right)^{b}$	[48]
Topp-Leone Weibull Distribution (TLWD)	$\left\{(1-e^{-\theta x^{\alpha}})\left[(2-(1-e^{-\theta x^{\alpha}})\right]\right\}^{b}$	[49]

Similar to the lifetime distributions generated using the Kumaraswamy framework, the TL-generated family of distributions listed in Table 3 also accommodate a left-skewed, right-skewed, symmetric shapes, increasing, decreasing, bathtub and inverted bathtub-shaped hazard property. The flexibility of the distributions in data fitting has been illustrated using real-life data sets across different fields of research.

2.4 The Continuous Bernoulli Distribution

The continuous Bernoulli distribution, an analogue to the Bernoulli distribution is apparently one of the most recent one-parameter continuous lifetime distributions with support [0,1]. The Continuous Bernoulli (CB) distribution with special application in machine learning has been introduced by [50]. The CB distribution is defined by the cumulative distribution function:

$$F(x,\alpha)=\begin{cases}\dfrac{\alpha^{x}(1-\alpha)^{1-x}+\alpha-1}{2\alpha-1}, & \alpha\neq \tfrac{1}{2}\\ x, & \alpha=\tfrac{1}{2}\end{cases} \tag{14}$$

with the associated probability density function obtained as

$$f(x,\alpha)=\begin{cases}C_{\alpha}\alpha^{x}(1-\alpha)^{1-x}, & \alpha\neq \tfrac{1}{2}\\ 1, & \alpha=\tfrac{1}{2}\end{cases}. \tag{15}$$

where C_{α} is referred to as the normalizing constant defined by

$$C_{\alpha}=\begin{cases}\dfrac{2\tanh^{-1}(1-2\alpha)}{1-2\alpha}, & \alpha\neq \tfrac{1}{2}\\ 2, & \alpha=\tfrac{1}{2}\end{cases} \tag{16}$$

since $\tanh^{-1}(x) = \frac{1}{2}\ln\left[\frac{1+x}{1-x}\right]$ and $2\tanh^{-1}(1-2\alpha) = \ln(1-\alpha) - \ln(\alpha)$.

A detailed study of some statistical properties of theContinuous Bernoulli distribution have been provided in [51]. In practice, the utility of the Continuous Bernoulli distribution over existing bounded distributions in modelling [0,1]-valued data set is arguably following its unique feature of possessing a continuous property and at some fixed value of the parameter, exhibits a discrete property. This is true, as most real data sets may look continuous in nature but by observation, discrete (countable).

It is essential to note that the CB distribution has received its first generalization in the work of [52]. By employing the Quadratic Rank Transmutation Map (QRTM) proposed by [53], they developed the Transmuted Continuous Bernoulli (TCB) distribution. The cdf and pdf of the TCB distribution are defined, respectively, by

$$F(x,\alpha,\lambda) = \begin{cases} 1-\left[1-\lambda\dfrac{\alpha^x(1-\alpha)^{1-x}+\alpha-1}{2\alpha-1}\right]\left[1-\dfrac{\alpha^x(1-\alpha)^{1-x}+\alpha-1}{2\alpha-1}\right], & \alpha \neq \tfrac{1}{2} \\ 1-(1-\lambda x)(1-x), & \alpha = \tfrac{1}{2} \end{cases}, \quad (17)$$

with the associated density function obtained as

$$f(x,\alpha,\lambda) = \begin{cases} C_\alpha \alpha^x (1-\alpha)^{1-x}\left[1+\lambda-2\lambda\dfrac{\alpha^x(1-\alpha)^{1-x}+\alpha-1}{2\alpha-1}\right], & \alpha \neq \tfrac{1}{2} \\ 1+\lambda-2x, & \alpha = \tfrac{1}{2} \end{cases}. \quad (18)$$

The authors illustrated the potentiality of the TCB distribution over the unit- Burr III, unit-Burr XII, and the classical Continuous Bernoulli distributions with three [0,1]-valued data sets.

3. Conclusion

In this paper, we have presented a detailed review on the generalization of [0,1]-valued lifetime distributions. Several generalized distributions developed based on the beta-G, Kumaraswamy-G, Topp-Leone-G and the recently introduced Continuous Bernoulli distribution were carefully highlighted.

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