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SOME SPECIAL CLASSES OF 3-PRIME NEAR-RINGS INVOLVING MULTIPLICATIVE DERIVATIONS

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ABSTRACT

This research work investigate some new results on near-rings through multiplicative derivations and present the commutativity of a 3-prime near-ring satisfying some differential and algebraic identities on nonzero Jordan ideals of 2-torsion free zero symmetric involving multiplicative derivations by considering two derivations instead of one derivation and established that if R is a 2-torsion free prime ring admitting a strong commutativity preserving (SCP) derivation d. Further, proved that if J is a nonzero Jordan ideal of a 2-torsion free zero symmetric together with 3-prime near-ring N and d_1, d_2 are two nonzero derivations on N such that d_2 is commuting on J then either $d_1 = 0$ on J or N is a multiplicative commutative near-ring and also prove some result on special class of near-rings with suitable constraints of its subsets.

1.0 INTRODUCTION

Throughout the paper a left near-ring $(N, +, \cdot)$ is a triplet, where *N* is a nonempty set together with two binary operations, addition $(+)$ and multiplication $(\cdot)$. The structure $(N.+)$ is a group, $(N, \cdot)$ is a semigroup, and $z\cdot(x+y) = z\cdot x + z\cdot y$ for all $x, y, z \in N$. As usual *N* represents $(N, +, \cdot)$. A near-ring *N* is said to be a zero-symmetric if $0x = 0$ for all $x \in N$ ($0x - 0$ for left distributive law). A near-ring *N* is a prime near-ring if for all $x, y \in N, xNy = (0)$ implies either $x = 0$ or $y = 0$ and *N* is semi prime near-ring if for $x \in N, xNx = (0)$ implies $x = 0$. *N* will be 3-prime near-ring if for all $x, y \in N$, $xNy = 0$ implies $x = 0$ or $y = 0$. *N* is called 2-torsion free if $2x = 0$ implies $x = 0$, for all $x \in N$. In general *N* is called *n*-torsion free if $nx = 0$ implies $x = 0$ for all $x \in N$. *Z(N)* is known as the multiplicative center of *N*, such that

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$Z=\{x\in N: xy=yx, \forall\ y\in N\}$. A near-ring N is called derivation on N for all $x, y\in N$ satisfies $d(xy)=d(x)y+xd(y)$ or $d(xy)=xd(y)+d(x)y$, $\forall x, y\in N$. The symbols $[x, y]=xy-yx$ and $(x\circ y)=xy+yx$ are both commutator (Lie product) and anticommutator (Jordan product) respectively. $[x, xy]=xxy-xyx=x(xy-yx)=x[x, y]$ and $(x\circ xy)=xxy+xyx=x(xy+yx)=x(x\circ y)$ are Jacobi Identities. The concept of derivation in rings is an essential and plays a major role in various branches of mathematics. The notion of derivation in rings established in 1957 by Posner [1] where he established two very striking results on derivations in prime rings. These results give an extensive importance in studying commutativity of rings, frequently prime ring and semiprime rings that admitting suitable constrained of derivations. Several authors have studies the prime rings and semiprime rings involving derivation in different directions. Bell and Mason [2] have been motivated by the concept of derivation in rings and introduced the notion of derivation in near-rings. Since then, many authors (see for example [3 – 9] for further reference) have investigate 3-prime near-ring and semiprime near-ring with 2-torsion free satisfying certain differential and algebraic identities involving derivations. The notion of multiplicative derivation was initiated by Daif [8], where the symbol $d: R\rightarrow R$ is called a multiplicative derivation if $d(xy)=xd(y)+d(x)y$ holds for all $x, y\in R.$Bell and Daif [7], established that if *R* is a 2-torsion free prime ring admitting a strong commutativity preserving (SCP) derivation *d*, i.e., *d* satisfies $[d(x), d(y)]=[x, y]$forevery $x, y\in R,$ then *R* is commutative. Further, Asma and Huque [3] proved that let *J* be a nonzero Jordan ideal of a 2-torsion free 3-prime near-ring *N*. If d_1, d_2 are two nonzero derivations on *N* such that d_2 is commuting on *J*and$[d_1(x), d_2(y)]=[x, y]$ for all $x\in J$ and $y\in N,$ then either $d_1=0$ on *J* or *N* is a commutative ring.

Motivated by these observations, it is a natural question to ask if *k* and *i* are positive integers instead of $k=0$ and $i=0$. In this works we give affirmative answer and extend the result of Asma and Huque [3] for a 3-prime near-ring involving two derivations. Motivated by Asma and Huque [3], Usman *et al*. [10] prove some result on special class of near-rings with suitable constraints of its subset. In addition, we prove the commutativity of a 3-prime near-ring *N* with its Jordan ideal.

2 Some Preliminary Results

In order to prove main results, we need the following lemmas.

2.1 Lemma

Let *J* be a nonzero Jordan ideal of a 2-torsion free 3-prime near-ring *N*. If $J\subseteq(N)$, then *N* is a commutative ring.

2.2 Lemma

Let *d* be a multiplicative derivation on a near ring *N*. Then

$$(d(x)y+xd(y))z=d(x)yz+xd(y)z,\quad \forall x, y\in N. \tag{1}$$

Proof. By solving $d(xyz)$ in two different ways, we have

$$d(xyz)=d(xy)z+xyd(z)=(d(x)y+xd(y))z+xyd(z) \tag{2}$$

and

$$d(xyz)=d(x)yz+xd(yz)=d(x)yz+xd(y)z+xyd(z) \tag{3}$$

Combining (2) and (3) we obtain

$$\left(d(x)y + xd(y)\right)z = d(x)yz + xd(y)z, \quad \forall x, y \in N. \qquad (4)$$

2.3 Lemma

Let N be a 3-prime near-ring.

i. If $z \in Z(N)\setminus 0$ and $xz \in Z(N)$, then $x \in Z(N)$.

ii. If $N \in Z(N)$, then N is a commutative ring.

iii. If N is 2-torsion free and d is a derivation on N such that $d^2 = 0$, then $d = 0$.

iv. If d is a derivation, then $x \in Z(N)$ implies $d(x) \in Z(N)$.

3 Some Results on 3-Prime Near-rings with Multiplicative Derivations

In this section we prove the commutativity of a 3-prime near-ring N with 2-torsion free involving multiplication derivations with its Jordan ideal.

3.1 Theorem

Let J be a nonzero Jordan ideal of a 2-torsion free 3-prime near ring N. If d is a multiplicative derivation on N satisfying one of the following conditions:

\begin{description}

(i) $d([x, y]) = x^k [x, y] x^i$, for all $x \in J$ and $y \in N$.

(ii) $d(x \circ y) = x^k [x, y] x^i$ for all $x \in J$ and $y \in N$.

(iii) $d([x, y]) = x^k (x \circ y) x^i$, for all $x \in J$ and $y \in N$.

where both k and i are integers greater than zero $(k > 0,$ and $i > 0)$. Then $d = 0$ on J or N is commutative.

Proof. (i) Our hypothesis

$$d([x, y]) = x^k [x, y] x^i, \quad \forall\ x \in J \text{ and } y \in N. \qquad (5)$$

Replace y by xy in equation (5) to get

$$d([x, xy]) = x^k [x, xy] x^i$$

$$d(x[x, y]) = x\left(x^k [x, y] x^i\right) \qquad (6)$$

substitute equation (5) in equation (6) to have

$$d(x[x, y]) = x\left(d[x, y]\right) \qquad (7)$$

Using the definition of the derivation on the above equation, we obtain

$$d(x)[x, y] + xd[x, y] = xd[x, y] \qquad (8)$$

$$d(x)[x, y] = 0 \qquad (9)$$

The above expression becomes

$$d(x)xy = d(x)yx, \ \forall x \in J,\ y \in N \qquad (10)$$

Putting yz instead of y in (10) and using it again, we get

$$d(x)xyz = d(x)yzx, \ \forall x \in J,\ y \in N$$

This reduces to

$$d(x)N[x, z] = \{0\}, \forall x \in J,\ z \in N$$

By 3-primeness of *N*, we get $d(x)=0$ or $x\in Z(N)$ for all $x\in J$. So from Lemma 2.3, (iv), the above two cases imply that $d(x)\in Z(N)$ and $d=0$ or *N* is commutative.

(ii) Our hypothesis

$d(x\circ y)=x^k[x, y]x^i$, $\forall\ x\in J$ and $y\in N$ (11)

Replace *y* by *xy* in equation (11) to get

$$d(x\circ xy)=x^k[x, xy]x^i,\ \forall\ x\in J \text{ and } y\in N$$

$$d(x(x\circ y))=x(x^k[x, y]x^i) \quad (12)$$

substitute equation (11) in equation (12) to have

$$d(x(x\circ y))=x(d(x\circ y)) \quad (13)$$

Using the definition of the derivation on the above equation, we obtain

$$d(x)(x\circ y)+xd(x\circ y)=xd(x\circ y) \quad (14)$$

$$d(x)(x\circ y)=0 \quad (15)$$

The above expression becomes

$$d(x)yx=-d(x)xy,\ \forall x\in J,\ y\in N \quad (16)$$

Putting *yz* instead of *y* in (16) and using it again, we have for all $x\in J,\ y\in N$

$$d(x)yzx=-d(x)xyz$$

$$d(x)yzx=d(x)xy(-z)$$

$$d(x)yzx=d(x)y(-x)(-z)$$

Putting $-x$ instead of *x* in the last expression, we obtain

$$d(-x)N[x, z]=\{0\},\ \forall x\in J,\ z\in N \quad (17)$$

By 3-primeness of *N*, we get $d(-x)=0$ or $x\in Z(N)$ for all $x\in J$. In the line of completion of this proof, we use the same way as used in case (i) to get the required result.

(iii) Our hypothesis

$$d([x, y])=x^k(x\circ y)x^i, \text{ for all } x\in J \text{ and } y\in N. \quad (18)$$

Replace *y* by *xy* in equation (18) to get

$$d([x, xy])=x^k(x\circ xy)x^i, \text{ for all } x\in J \text{ and } y\in N.$$

$$d(x[x, y])=x(x^k(x\circ y)x^i) \quad (19)$$

substitute equation (18) in equation (19) to get

$$d(x[x, y])=x(d[x, y]) \quad (20)$$

Using the definition of the derivation on the above equation, we obtain

$$d(x)[x, y]+xd[x, y]=xd[x, y]$$

$$d(x)[x, y]=0 \quad (21)$$

The above expression becomes

$$d(x)yx=d(x)xy,\ \forall x\in J,\ y\in N \quad (22)$$

Putting *yz* instead of *y* in (22) and using it again, we get

$d(x)yzx = d(x)xyz, \ \forall x \in J, \ y \in N$

This reduces to

$d(x)N[x, z] = \{0\}, \ \forall x \in J, \ z \in N$

By 3-primeness of *N*, we get $d(x) = 0$ or $x \in Z(N)$ for all $x \in J$. As case (i) repeat the same line to obtain the required result.

3.2 Theorem

Let *J* be a nonzero Jordan ideal of a 3-prime near ring *N*. If *d* is a multiplicative derivation on *N* satisfying one of the following Conditions:

i. $d(x)d(y) = y^k[y, x]y^i$

ii. $d(x)d(y) = y^k(y \circ x)y^i$, for all $y \in J$ and $x \in N$, then $J \subseteq Z(N)$.

Proof. (i) By hypothesis

$d(x)d(y) = y^k[y, x]y^i, \ \forall y \in J, \ x \in N$ (23)

Replacing *x* by *yx* in equation (23) to get

$d(yx)d(y) = y(y^k[y, x]y^i)$ (24)

By definition of derivation the above equation becomes

$(yd(x) + d(y)x)d(y) = y(y^k[y, x]y^i)$

$yd(x)d(y) + d(y)xd(y) = y(y^k[y, x]y^i)$ (25)

substitute equation (23) in to equation (24) to obtain

$yd(x)d(y) + d(y)xd(y) = yd(x)d(y)$

Applying Lemma 2.2. The above equation reduce to

$d(y)xd(y) = 0, \ \forall y \in J$ and $x \in N$ (26)

The above equations becomes

$d(y)Nd(y) = 0, \ \forall y \in J$ and $x \in N$ (27)

Since *N* is 3-prime, we find that $d(y) = 0$ for all $y \in J$. Therefore, our hypothesis gives $xy = yx$ for all $y \in J$ and $x \in N$. Putting *xz* instead of *x*, we have $x[y, z] = 0$ for all $y \in J$ and $x, z \in N$ Replacing *x* by $[y, z]x$ in the equation (27) and using the 3-primeness of *N*, Hence, by Lemma 2.1, the last expression gives either $d(y) = 0$ for all $y \in J$ or *N* is a commutative ring.

(ii) By hypothesis,

$d(x)d(y) = y^k(y \circ x)y^i, \ \forall \ y \in J$ and $x \in N$ (28)

Replacing *x* by *yx* in equation (28) to get

$d(yx)d(y) = y(y^k(y \circ x)y^i)$ (29)

By definition of derivation the above equation becomes

$(yd(x) + d(y)x)d(y) = y(y^k(y \circ x)y^i)$

$yd(x)d(y) + d(y)xd(y) = y(y^k(y \circ x)y^i)$ (30)

substitute equation (28) in to equation (29) to obtain

$$yd(x)d(y)+d(y)xd(y)=yd(x)d(y)$$

Applying Lemma 2.2. The above equation become

$$d(y)xd(y)=0,\ \forall y\in J,\ and\ \ x\in N \tag{31}$$

Since N is 3-prime, we find that $d(y)=0$ for all $y\in J$. Therefore, our hypothesis gives $yx=-xy$ for all $y\in J$. and $x\in N$. Putting xz instead of x, we have $x[y,z]=0$ for all $y\in J$. and $x, z\in N$. Replacing x by $[y,z]x$ in the previous expression and using the 3-primeness of N. Hence, by Lemma 2.1, the last expression gives either $d(y)=0$ for all $y\in J$. or N is a commutative ring.

4 Conclusion

This research work deals some new results on rings and near-rings through derivations, multiplicative derivations with their fundamentals. Firstly, it is shown that some commutativity results for rings and near-rings with algebraic identities of Lie and Jordan products involving multiplicative derivations. Secondly, the certain results presented in this work are extension of previously, obtained results and also prove some result on special class of rings and near-rings with suitable constraints of its subsets via derivations, multiplicative derivations. Thirdly, it is prove that the commutativity of prime near-ring satisfying the differential identities on Jordan ideals involving multiplicative derivations. Finally, we improve and extend some recent results on 3-prime near-rings via multiplicative derivation.

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