Transactions of
The Nigerian Association of
Mathematical Physics
Journal homepage: https://nampjournals.org.ng

THE DYNAMIC BUCKLING LOAD FORMULA OF A CUBIC MODEL STRUCTURE STRUCK BY SLOWLY VARYING FREQUENCY

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ARTICLE INFO

Article history:
Received xxxxx
Revised xxxxx
Accepted xxxxx
Available onlinexxxxx

Keywords:
Buckling,
Asymptotic,
Perturbation,
Static,
Dynamic.

ABSTRACT

This investigation is centred on an analytical determination of the dynamic buckling loads formula of some viscously damped elastic structures pressurized by a periodic load with slowly varying circular frequencies. The formulation has two small but mathematically independent parameters which allow the use of asymptotic expansions of the variables. In addition, a two-timing regular perturbation procedure is used to analyze the relevant equations which contain some degrees of nonlinearities in their formulation. The dynamic buckling load, λ_D ***lies between 0 and 1,*** $0 < \lambda_D << 1$***.***

1.1 BACKGROUND OF STUDY

In this study, we are aiming at finding the dynamic buckling loads of some imperfect elastic structures having light viscous damping but trapped by a periodic load with slowly varying circular frequency. The analysis is based on already existing theory derived by [1] and [2]. The two structures studied by [3] are the simple cubic elastic model structure and an imperfect elastic spherical cap whose normal displacement is regarded as the summation of the modes, that is, pre-buckling, an axis symmetric and non-axis symmetric modes, all having time dependent amplitudes. The theory is here developed through the simple cubic elastic model structure followed by a practical application on an elastic spherical cap, where all deformations (failures) are controlled within elastic range. The problems derived are non-linear which need to be solved but there is no simple analytical method to handle the solutions. We choose perturbation method to handle the problem because of the presence of two small mathematically independent parameters through which a two-timing perturbation scheme is formed. In the same vein, the problem now becomes a two–small parameter non-linear problem, which the solution here can be approached analytically using regular perturbation and asymptotic methods.

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https://www.doi.org/10.60787/tnamp-19-185-206

In most countries of the world we often hear of collapse of buildings, bridges and other material structures. These are forms of material failures which are dangerous in nature and should be prevented by all cost. Great investigations and efforts have been done by Engineers and Applied Mathematicians to determine the maximum loads that structures can carry before buckling occurs but still, buckling of elastic structures still comes to play from time to time. The dynamic buckling analysis of imperfect spherical shell with light viscous damping trapped by a periodic load with slowly varying frequency, is a real life problem, yet, it does not seem, to our knowledge, that a lot of analytical investigations have been done on the subject matter. Most of the earlier works done on this area, used numerical methods (Finite Element methods) for the analysis of buckling. In this study, we are carrying out investigations using purely analytical methods to investigate the dynamic buckling and stability of two viscously damped elastic structures, namely, a simple cubic model and an imperfect elastic spherical shell pressurized by a periodic load that has a slowly varying frequency. In addition, we investigate the effect of light viscous damping on the dynamic stability of the structures.

The concept of dynamic buckling is presently an interesting area of research which many researchers have gone into based on the reality that it is closely related to some other fields of human endeavor. Structural elastic materials have the tendencies of undergoing deformations and other instabilities when loaded either statically or dynamically. One of the major concerns of structural Engineers and Applied Mathematicians is to know the load carrying capacity of a given elastic material before buckling. Many researchers have worked on dynamic stability of structures by subjecting these materials to various loading conditions .We recall that [1] studied the step loading, impulse loading, rectangular loading and triangular loading while [2] analyzed the impact of periodic loading on elastic structures. Based on researches conducted by many investigators some of which are [1],[2] and others, it has become clear that initial imperfections, the loading history (i.e. the nature of the loading), the time duration and the elastic characteristics of materials affect the dynamic buckling loads of structures. To our knowledge, periodic loading with slowly varying circular frequency is not commonly discussed in the area of buckling. It is however quite pertinenent to mention that the concept of periodic forcing with slowly varying frequency in the time variables was discussed by [4]. In that study, the frequency was assumed to have a slow cubic variation with time and was discussed in the context of a weakly damped and weakly non-linear excitation involving Duffing's equation.

Currently, most studies on dynamic buckling appear to centre on beams, plates, columns, spherical shells and cylindrical shells with extensive literatures and the techniques adopted are mostly numerical approach. In view of this, mention must be made of [5], who studied the dynamic buckling of thin – walled viscoplastic columns while [6], similarly studied some aspects of dynamic buckling of plates under in-plane pulse compression. In the same way, [7], studied some important parameters in dynamic buckling analysis of plated structures subjected to impulse loading, [8], investigated Asymptotic investigation of the buckling of a cubic-quintic nonlinear elastic model structure stressed by static load and a dynamic step load, [9] studied the imperfect Bifurcation with a slowly – vary control parameter while [2], studied the buckling of impulsively loaded prismatic cores. Worthy of mentioning are [10], who investigated the stability of transverse vibration of rod under longitudinal step wise loading, [11] who investigated advances in shell buckling theory and experiments, [2], studied dynamic buckling estimates and [12] who investigated the influence of uncertainties on the dynamic buckling loads of structures liable to Asymmetric post buckling behavior. The study of buckling behavior of beams and columns was studied by [13], a study on buckling – waves was carried out by [14] while [15] dealt on the dynamic buckling of a model structure with quadratic non-linearity struck by a step load superposed on Quasi- static load. The investigation into the dynamic effects of lateral buckling of high temperature/High pressure offshore pipelines was carried out by [16], while [17] investigated the dynamic buckling of shallow pin-ended arches under a sudden central concentrated load. In the same manner, [18] carried out investigation on elastic buckling of steel columns under axial compression, [19] studied elastic buckling of columns with end restraint effects, [20] conducted an investigation on analytic approach for exactly determining critical loads of buckling of non-uniform columns while [21], investigated the buckling of variables section columns under axial loading.

The following investigations are equally of relevance in this work : [22] who studied perturbation technique in the buckling of some elastic materials struck by a periodic load with slowly varying frequency in which this work is an obvious extension by assuming light viscous damping, [23] investigated the dynamics of rods under axial impact, [24] embarked on numerical methods for determining strongest cantilever beam with constant volume, [25] studied nonlinear stochastic dynamical post buckling analysis of uncertain cylindrical shells, [26] investigated dynamic buckling of composite cylindrical shell subjected to axial impulse while [27] conducted a research on numerical and experimental stability of buckling of advanced fibre composite cylinders under axial compression.In the same vein,

[28] investigated dynamic buckling of an inclined structure, [29] studied the dynamic buckling of thin thermovisplastic rectangular plate, [30] researched on the buckling of a clamped viscously damped column trapped by a step load, [39] researched on the buckling load of elastic quadratic non-linear structures by an axial impulse while [31] worked on analysis of tilt- buckling of Euler columns with varying flexural stiffness using homotopy perturbation method.

Furthermore, [32] made remarkable contributions to the field on vibration analysis of cracked frame structures, [33] conducted research on the comparison of critical buckling load in regression, fuzzy logic and ANN based estimations while [34] studied buckling analysis of a beam-column using multilayer perception neural network technique. We note that [35] investigated the determination of buckling loads and mode shapes of heavy vertical column under its own weights using variational iteration method , [36] carried out a research on buckling of axially loaded castellated steel columns while [37] made a contribution on a simple method to determine the critical loads for axially in- homogenous beams with elastic restraints, [38] carried out an investigation on the buckling of Euler columns with a continuous elastic restraint via homotopy analysis method, [39] gave the solution for the problem of a new FEM procedure for transverse and longiditunal vibration analysis of thin rectangular plates subjected to a variable velocity moving load along an arbitrary trajectory and [40] investigated the asymptotic analysis of an improved quadratic model structure subjected to static loading. In the same way, [41] investigated the shape optimization of damaged columns subjected to conservative and non-conservative forces, [42] made investigation on maximum load factor corresponding to a slightly asymmetric bifurcation point, [43] studied an analytical formulation for local buckling and postbuckling analysis of stiffened laminated panels, [44] investigated the optimal design of clamped columns for stability under combined axial compression and torsion while [8] studied the asymptotic analysis of the static buckling of infinitely long and harmonically imperfect column lying on quadratic- cubic elastic foundations.

2.0 METHODOLOGY

The methodology applied in this work is anchored on regular perturbation and asymptotic analysis. The type of problem solved automatically demands the use of these methods. The two problems confronted in this work are non-linear and so, we cannot determine their exact analytical solutions in a closed form. The formulations in both cases have two small mathematically independent parameters in which asymptotic series expansions are used in this work. To get a uniformly valid asymptotic solution in each case, we shall apply two-timing regular perturbation techniques in the two problems. Expansions are asymptotic and are valid in the limit as the two small parameters become very small compared to unity. Usually, in most multi-timing perturbation problems, the use of two timing procedure converts the ordinary differential equations in which the problems were originally posed, to partial differential equations.

In each of these two problems, our initial intention is to get the displacement, and later, the maximum displacement, η_a. The condition for buckling (i.e. at maximum displacement), as in [2] is

$$\frac{d\lambda}{d\eta_a} = 0 \qquad (2.1)$$

This is to be evaluated to get the dynamic buckling, λ_D.

The dynamic buckling load λ_D is the largest load parameter for the problem to have a bounded solution.

2.1 FORMULATION OF THE SIMPLE MODEL PROBLEM

DETERMINATION OF THE DYNAMIC BUCKLING LOAD IN THE CASE OF CUBIC MODEL STRUCTURE

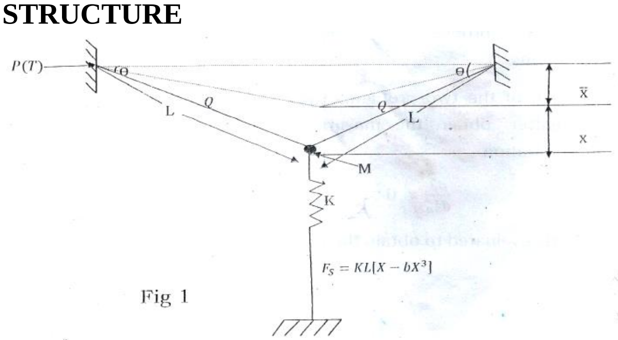

Fig 1

SIMPLE CUBIC MODEL STRUCTURE

We consider a simple model in the form of two bars of equal length L which fixed at both ends to the rigid wall as shown in the figure 1 above. The bars are subjected to a horizontal load P(T) applied shortly after time T =0. The bar must be rigid to avoid deformation when load is applied. A mass M is suspended at the meeting point of the two arms of the bars whose movement is regulated by a non-linear (cubic) spring that gives a restoring force F_s per unit length of KL ($X-bX^3$), b>0 where K is a spring constant, where X is the additional displacement from the point of equilibrium. Let $\bar{X}$ be the imperfection parameter when the two arms of the bars are joined in the horizontal direction. Let Q be the tension (force) on the each arm of the bars and θ be the angle between the horizontal direction and each arm of the bars.

We assume that θ must be small relative to unity .Therefore we make the following approximations

$$Sin\,\theta = \frac{\bar{X} + X}{L}, \qquad \sin\theta \cong \theta$$

$\therefore\ \theta = \frac{\bar{X}+X}{L}$ (2.2)

It is to be noted that P(T) on the horizontal axis relates with the tension Q on each arm of the bars. For equilibrium of the forces on the axial direction where P(T) acts, we obtain

$Cos\,\theta = \frac{P(T)}{Q}, \mathrm{P(T)} = \mathrm{Q}Cos\,\theta$ (2.3a,b)

For equilibrium of forces in the vertical direction, where mass M acts, we have

$F_y = 2Q\ Sin\theta$ (2.4)

The restoring force of the spring is,

$$F_s = KL\left(\frac{X}{L} - b\left(\frac{X}{L}\right)^3\right) \tag{2.5}$$

The net vertical force is,

$F = F_y - F_s$ (2.6)

According to Newton's law of motion

$F = Ma = \frac{M}{L}\frac{d^2X}{dT^2}$ (2.7a)

$where\ a\ is\ the\ acceleration$, hence, (3.7a) becomes,

$\frac{M}{L}\frac{d^2X}{dT^2} = F$ (2.7b)

Substituting (2.4), (2.5) in (2.6), gives,

$F = 2QSin\theta - KL\left(\frac{X}{L} - b\left(\frac{X}{L}\right)^3\right)$ (2.8)

Hence, substituting (2.8) in (2.7b), gives,

$\frac{M}{L}\frac{d^2X}{dT^2} = 2QSin\theta - KL\left(\frac{X}{L} - b\left(\frac{X}{L}\right)^3\right)$ (2.9)

Then, (2.3b), becomes,

$P(T) = Q$ (2.10)

Substituting (2.2) and (2.10) in (2.9), gives

$$\frac{M}{L}\frac{d^2X}{dT^2} = 2P(T)\left(\frac{\bar{X}+X}{L}\right) - KL\left(\frac{X}{L} - b\left(\frac{X}{L}\right)^3\right) = 2P(T)\frac{\bar{X}}{L} + 2P(T)\frac{X}{L} - KL\frac{X}{L} + bKL\left(\frac{X}{L}\right)^3$$

$$= 2P(T)\frac{\bar{X}}{L} + KL\frac{X}{L}\left(\frac{2P(T)}{KL^2} - 1\right) + bKL\left(\frac{X}{L}\right)^3 \tag{2.11}$$

$$From\ (2.11), we\ make\ 2P(T)\frac{\bar{X}}{L}\ the\ subject\ and\ get$$

$\frac{M}{L}\frac{d^2X}{dT^2} + KL\left(1 - \frac{2P(T)}{KL^2}\right)\frac{X}{L} - bKL\left(\frac{X}{L}\right)^3 = \frac{2P(T)\bar{X}}{L}$ (2.12)

Now it is pertinent to introduce the following non-linear dimensional equations,

$\eta = \frac{X}{L};\ \epsilon = \frac{\bar{X}}{L}; t = T\sqrt{\frac{KL}{M}},\ \ 0 < \epsilon \ll 1$ (2.13)

Let,

$$2P(T) = \frac{2P(T)P(0)}{P(0)} = 2f(t)P(0) \quad (2.14)$$

where,

$$f(t) = \frac{P(T)}{P(0)};\ \lambda = \frac{2P(0)}{KL^2}; P(0) \neq 0 \quad (2.15)$$

Substituting (2.13), (2.14) and (2.15) in (2.12), the results gives

$$M\left(\frac{KL^2}{M}\right)\frac{d^2\eta}{dt^2} + KL\left(1 - \frac{2f(t)P(0)}{KL^2}\right)\eta - KLb(\eta)^3 = 2\epsilon f(t)P(0)$$

This implies that,

$$KL^2\frac{d^2\eta}{dt^2} + KL^2\left(1 - \frac{2f(t)P(0)}{KL^2}\right)\eta - KbL^2\eta^3 = 2\epsilon f(t)P(0) \quad (2.16)$$

After simplifying (2.16) using (2.13)-(2.15), (2.16) becomes,

$$\frac{d^2\eta}{dt^2} + \big(1 - \lambda f(t)\big)\eta - b\eta^3 = \lambda\epsilon f(t); t > 0 \quad (2.17a)$$

$$\eta(0) = \frac{d\eta(0)}{dt} \quad (2.17b)$$

$Equation(2.17a)\ is\ the\ required\ equation\ of\ motion\ for\ all\ types\ of\ loading \lambda f(t)$. It was first derived by [2]

For our work, $\lambda f(t)$ is periodic load with slowly varying frequency with amplitude λ. We let,

$$f(t) = Cos\,(\omega(\delta t)); 0 < t < \infty; 0 < \delta \ll 1 \quad (2.18)$$

$where\ \omega(\delta t) is\ a\ continuous\ slowly\ varying\ time\ dependent\ frequency\ function$
$with\ right\ hand\ derivatives\ of\ all\ orders\ at\ t = 0\ and\ is\ such\ that$
$\omega(0) = 0;\ |\omega(\delta t)| \ll 1; 0 < \delta << 1.$

Then, the relevant equation of motion (2.17a), which is here adjusted to include a light viscous damping now becomes

$$\frac{d^2\eta}{dt^2} + 2\delta\frac{d\eta}{dt} + (1 - \lambda\cos\,(w(\delta t))\eta - b\eta^3 = \lambda\epsilon\cos\big(w(\delta t)\big) \quad (2.19a)$$

where b is the imperfection - sensitivity parameter and $2\delta\frac{d\eta}{dt}$ is the damping term with coefficient 2δ.The damping is said to be light because of the coefficient 2δ is such that $0 < \delta << 1$.

We now let,

$$\tau = \delta t \quad (2.19b)$$

$$\therefore \omega = \omega(\delta t) = \omega(\tau) \quad (2.20a)$$

The general equation (2.19a) forms a second order non-linear, non-homogenous differential equation with slowly varying time-dependent and periodic co-efficient where τ is a slow time scale and δ is small compared to unity .

Here,$\lambda\ is\ load\ parameter, for\ 0 < \lambda < 1, \eta\ isthe\ deflection(displacement)$

Now, we let,

$$\frac{d\tilde{t}}{dt} = [1 - \lambda\cos\,(\omega(\delta t))]^{1/2} = [1 - \lambda\cos\,(\omega(\tau))]^{1/2} \quad (2.20b)$$

$$\hat{t} = \tilde{t} + \frac{1}{\delta}\left\{\sum_{i=1}^{n}\mu_i(\tau)\epsilon^i\right\} = \tilde{t} + \frac{1}{\delta}[\mu_1(\tau)\epsilon + \mu_2(\tau)\epsilon^2 + \cdots] \quad (2.20c)$$

$where,$

$\mu_i(0) = 0; i = 1,2,3, ...$

Therefore, we have,

$$\frac{d\eta}{dt} = \frac{\partial\eta}{\partial\hat{t}}\frac{\partial\hat{t}}{\partial\tilde{t}}\frac{\partial\tilde{t}}{\partial t} + \frac{\partial\eta}{\partial\hat{t}}\frac{\partial\hat{t}}{\partial\tau}\frac{d\tau}{dt} + \frac{\partial\eta}{\partial\tau}\frac{d\tau}{dt} \quad (2.21a)$$

We note that,

$$\frac{\partial\eta}{\partial\hat{t}} = \eta_{,\hat{t}}\ ;\ \frac{\partial\hat{t}}{\partial\tilde{t}} = 1;\ \frac{\partial\eta}{\partial\tau} = \eta_{,\tau}\ ;\ \frac{d\tau}{dt} = \delta$$

$$\frac{\partial\hat{t}}{\partial\tau} = \frac{1}{\delta}(\mu_1'\,\epsilon + \mu_2'\epsilon^2 + \mu_3'\epsilon^3 + \cdots) \quad (2.21b)$$

Here a subscript following a comma indicates partial differentiation and

$$\frac{d(\ldots)}{d\tau} = (\ldots)'$$

Substituting (2.21b) in (2.21a), we obtain,

$$\frac{d\eta}{dt} = (1-\lambda\cos(\omega(\tau))^{1/2}\,\eta_{,\hat{t}} + \frac{1}{\delta}\left(\mu_{,}'\epsilon + \mu_2'\epsilon^2 + \mu_3'\epsilon^3 + \cdots\right)\delta\eta_{,\hat{t}} + \delta\eta_{,\tau}$$

$$= (1-\lambda\cos(\omega(\tau))^{1/2}\,\eta_{,\hat{t}} + (\mu_1'\epsilon + \mu_2'\epsilon^2 + \mu_3'\epsilon^3 + \cdots)\eta_{,\hat{t}} + \delta\eta_{,\tau} \qquad (2.21c)$$

Hence forth, we shall write $\cos(\omega(\tau))$ *simply as* $\cos(\omega)$

Then, it follows that,

$$\frac{d^2\eta}{dt^2} = (1-\lambda\cos(\omega))\eta_{,\hat{t}\hat{t}} + 2(1-\lambda\cos(\omega))^{1/2}(\mu_1'\epsilon + \mu_2'\epsilon^2 + \cdots)\eta_{,\hat{t}\hat{t}}$$

$$+ (\mu_1'\epsilon + \mu_2'\epsilon^2 + \cdots)^2\eta_{,\hat{t}\hat{t}} + 2\delta(1-\lambda\cos(\omega))^{1/2}\eta_{,\hat{t}\tau} + 2\delta(\mu_1'\epsilon + \mu_2'\epsilon^2 + \mu_3'\epsilon^3 + \cdots)\eta_{,\hat{t}\tau}$$

$$+ \frac{\delta\lambda\omega'\sin(\omega)}{2(1-\lambda\cos(\omega))^{1/2}}\eta,\hat{t} + \delta^2\eta_{,\tau\tau} + \left(\mu_1''\epsilon + \mu_2''\epsilon^2 + \mu_3''\epsilon^3 + \cdots\right)\eta_{,\hat{t}} \qquad (2.22a)$$

It is to be recalled that the relevant equation is,

$$\frac{d^2\eta}{dt^2} + 2\delta\frac{d\eta}{dt} + (1-\lambda\cos(\omega))\eta - b\eta^3 = \lambda\epsilon\cos(\omega) \qquad (2.22b)$$

Substituting (2.21c) and (2.22a) in (2.22b), leads to,

$$\Big[(1-\lambda\cos(\omega))\eta_{,\hat{t}\hat{t}} + 2(1-\lambda\cos(\omega))^{1/2}(\mu_1'\epsilon + \mu_2'\epsilon^2 + \mu_3'\epsilon^3 + \cdots)\eta_{,\hat{t}\hat{t}}$$

$$+(\mu_1'\epsilon + \mu_2'\epsilon^2 + \mu_3'\epsilon^3 + \cdots)^2\eta_{,\hat{t}\hat{t}} + 2\delta(1-\lambda\cos(\omega))^{1/2}\eta_{,\hat{t}\tau} + 2\delta(\mu_1'\epsilon + \mu_2'\epsilon^2$$

$$+\mu_3'\epsilon^3 + \cdots)\eta_{,\hat{t}\tau} + \frac{\lambda\delta\omega'\sin(\omega)}{2(1-\lambda\cos(\omega))^{1/2}}\eta_{,\hat{t}} + \delta^2\eta_{,\tau\tau} + \left(\mu_1''\epsilon + \mu_2''\epsilon^2 + \mu_3''\epsilon^3 + \cdots\right)\eta_{,\hat{t}}\Big]$$

$$+2\delta\left[(1-\lambda\cos(\omega))^{1/2}\eta_{,\hat{t}} + (\mu_1'\epsilon + \mu_2'\epsilon^2 + \mu_3'\epsilon^3 + \cdots)\eta_{,\hat{t}} + \delta\eta_{,\tau}\right]$$

$$+(1-\lambda\cos(\omega))\eta - b\eta^3 = \lambda\epsilon\cos(\omega)(2.23)$$

Dividing (2.23) *through by* $(1-\lambda\cos(\omega))$ *gives,*

$$\Big[\eta_{,\hat{t}\hat{t}} + \frac{2}{1-\lambda\cos(\omega)^{1/2}}(\mu_1'\epsilon + \mu_2'\epsilon^2 + \mu_3'\epsilon^3 + \cdots)\eta_{,\hat{t}\hat{t}} + \frac{1}{(1-\lambda\cos(\omega))}(\mu_1'\epsilon + \mu_2'\epsilon^2 + \mu_3'\epsilon^3 + \cdots)\eta_{,\hat{t}\hat{t}}$$

$$+\frac{2\delta}{(1-\lambda\cos(\omega))^{1/2}}\eta_{,\hat{t}\tau} + \frac{2\delta}{(1-\lambda\cos(\omega))}(\mu_1'\epsilon + \mu_2'\epsilon^2 + \mu_3'\epsilon^3 + \cdots)\eta_{,\hat{t}\tau}$$

$$+\frac{\lambda\delta\omega'\sin(\omega)\,\eta_{,\hat{t}}}{2(1-\lambda\cos(\omega))^{3/2}} + \frac{\delta^2}{(1-\lambda\cos(\omega))}\eta_{,\tau\tau} + \frac{1}{(1-\lambda\cos(\omega))}\left(\mu_1''\epsilon + \mu_2''\epsilon^2 + \mu_3''\epsilon^3 + \cdots\right)\eta_{,\hat{t}}\Big]$$

$$+\left[\frac{2\delta}{(1-\lambda\cos(\omega))^{1/2}}\eta_{,\hat{t}} + \frac{2\delta}{(1-\lambda\cos(\omega))}(\mu_1'\epsilon + \mu_2'\epsilon^2 + \mu_3'\epsilon^3 + \cdots)\eta_{,\hat{t}} + \frac{2\delta^2}{(1-\lambda\cos(\omega))}\eta_{,\tau}\right]$$

$$+\eta - \frac{b}{(1-\lambda\cos(\omega))}\eta^3 = \frac{\lambda\epsilon\cos(\omega)}{(1-\lambda\cos(\omega))} \qquad (2.24)$$

Now it is pertinent to assume the following asymptotic series

$$\eta(\hat{t},\tau) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\eta^{ij}(\hat{t},\tau)\,\epsilon^i\delta^j(2.25)$$

Where ij on η^{ij} indicates superscript but not powers

On expansion, (2.25) *leads to,*

$$\eta(\hat{t},\tau) = \epsilon\,(\eta^{10} + \delta\eta^{11} + \delta^2\eta^{12} + \cdots) + \epsilon^2(\eta^{20} + \delta\eta^{21} + \delta^2\eta^{22} + \cdots)$$

$$+\epsilon^3(\eta^{30} + \delta\eta^{31} + \delta^2\eta^{32} + \cdots) + \cdots \qquad (2.25)$$

Equating equations of orders $(\epsilon^i\delta^j)$ *in* (2.24) *using* (2.25) *and* (2.26), gives the following equations,

$$O(\epsilon): \eta^{10}_{,\hat{t}\hat{t}} + \eta^{10} = \frac{\lambda\cos(\omega)}{1-\lambda\cos(\omega)} \qquad (2.27a)$$

$$O(\epsilon\delta): \eta^{11}_{,\hat{t}\hat{t}} + \eta^{11} = \frac{-2\eta^{10}_{,\hat{t}\hat{t}}}{(1-\lambda\cos(\omega))^{1/2}} - \frac{2\eta^{10}_{,\hat{t}\hat{\tau}}}{(1-\lambda\cos(\omega))^{1/2}} - \frac{\lambda\omega'\sin(\omega)\eta^{10}_{,\hat{t}}}{2(1-\lambda\cos(\omega))^{3/2}} \quad (2.27b)$$

$$O(\epsilon\delta^2): \eta^{12}_{,\hat{t}\hat{t}} + \eta^{12} = \frac{-2\eta^{10}_{,\tau}}{(1-\lambda\cos(\omega))^{1/2}} - \frac{\eta^{10}_{.\tau\tau}}{(1-\lambda\cos(\omega))}\frac{-2\eta^{11}_{,\hat{t}\tau}}{(1-\lambda\cos(\omega))^{1/2}} - \frac{2\eta^{11}_{,\hat{t}}}{(1-\lambda\cos(\omega))^{1/2}} - \frac{\lambda\omega'\sin(\omega)\eta^{11}_{,\hat{t}}}{2(1-\lambda\cos(\omega))^{3/2}} \quad (2.27c)$$

$$O(\epsilon^3): \eta^{30}_{,\hat{t}\hat{t}} + \eta^{30} = \frac{b(\eta^{10})^3}{(1-\lambda\cos(\omega))} - \frac{2\mu_2'\eta^{10}_{,\hat{t}\hat{t}}}{(1-\lambda\cos\ \omega))} \quad (2.27d)$$

$$O(\epsilon^3\delta): \eta^{31}_{,\hat{t}\hat{t}} + \eta^{31} = \frac{3b(\eta^{10})^2\eta^{11}}{(1-\lambda\cos(\omega))} - \frac{2\mu_2'\eta^{11}_{,\hat{t}\hat{t}}}{(1-\lambda\cos(\omega))^{1/2}} - \frac{2\mu_2'\eta^{30}_{,\hat{t}}}{(1-\lambda\cos(\omega))^{1/2}} - \frac{2\eta^{30}_{,\hat{t}\tau}}{(1-\lambda\cos(\omega))^{1/2}} - \frac{2\mu_2'\eta^{10}_{,\hat{t}\tau}}{(1-\lambda\cos(\omega))} - \frac{2\mu_2'\eta^{10}_{,\hat{t}}}{(1-\lambda\cos(\omega))} - \frac{\mu_2''\eta^{11}_{,\hat{t}}}{(1-\lambda\cos(\omega))} - \frac{\lambda\omega'\sin(\omega)\eta^{30}_{,\hat{t}}}{2(1-\lambda\cos(\omega))^{3/2}} \quad (2.27e)$$

$$O(\epsilon^3\delta^2): \eta^{32}_{,\hat{t}\hat{t}} + \eta^{32} = \frac{3b}{(1-\lambda\cos(\omega))}\{\eta^{10}(\eta^{11})^2 + (\eta^{10})^2\eta^{12}\} - \frac{2\mu_2'\eta^{12}_{,\hat{t}\hat{t}}}{(1-\lambda\cos(\omega))^{1/2}} - \frac{2\eta^{31}_{,\hat{t}\tau}}{(1-\lambda\cos(\omega))^{1/2}} - \frac{\eta^{31}_{,\hat{t}}}{(1-\lambda\cos(\omega))^{1/2}} - \frac{2\mu_2^1\eta^{11}_{,\hat{t}\tau}}{(1-\lambda\cos(\omega))} - \frac{2\mu_2^1\eta^{11}_{,\hat{t}}}{(1-\lambda\cos(\omega))} - \frac{\eta_2^{11}\eta^{11}_{,\hat{t}}}{(1-\lambda\cos(\omega))} - \frac{2\eta^{30}_{,\tau}}{(1-\lambda\cos(\omega))} - \frac{\eta^{30}_{,\tau\tau}}{(1-\lambda\cos(\omega))} - \frac{\lambda\omega'\sin(\omega)\eta^{31}_{,\hat{t}}}{2(1-\lambda\cos(\omega))^{3/2}} \quad (2.27f)$$

The initial conditions are,

$$\eta(0) = \frac{d\eta(0)}{dt} = 0 \quad (2.28a)$$

However, realizing the fact that,

$$\frac{d\eta}{dt} = (1-\lambda\cos(\omega))^{1/2}\eta_{,\hat{t}} + (\mu_2'(\tau)\epsilon^2 + \mu_3'(\tau)\epsilon^3 + \cdots)\eta_{,\hat{t}} + \delta\eta_{,\tau} \quad (2.28b)$$

the velocity initial condition $\frac{d\eta}{dt} = 0$ *in* $(2.28b)$ *becomes,*

$$\eta_{,\hat{t}} + (1-\lambda\cos(\omega))^{-1/2}\{\mu_2'\epsilon^2 + \mu_3'\epsilon^3 + \cdots\}\eta_{,\hat{t}} + \delta(1-\lambda\cos(\omega))^{-1/2}\eta_{,\tau} = 0 \quad (2.29)$$

Taking the initial conditions, (2.29) *results to*

$$\eta_{,\hat{t}}(0,0) + (1-\lambda)^{-1/2}\{\mu_2'\epsilon^2 + \mu_3'\epsilon^3 + \cdots\}\eta_{,\hat{t}}(0,0) + \delta(1-\lambda)^{-1/2}\eta_{,\tau}(0,0) = 0 \quad (2.30)$$

Other initial conditions in orders of $(\epsilon^i\delta^j)$ are,

$$\eta^{ij}(0,0) = 0\ \forall\ i,j \quad (2.31a)$$

$$O(\epsilon): \eta^{10}(0,0) = \eta^{10}_{,\hat{t}}(0,0) = 0 \quad (2.31b)$$

$$O(\epsilon\delta): \eta^{11}_{,\hat{t}}(0,0) + (1-\lambda)^{-1/2}\eta^{10}_{,\tau} = 0 \quad (2.31c)$$

$$O(\epsilon\delta^2): \eta^{12}_{,\hat{t}}(0,0) + (1-\lambda)^{-1/2}\eta^{11}_{,\tau} = 0 \quad (2.31d)$$

$$O(\epsilon^3): \eta^{30}_{,\hat{t}}(0,0) + (1-\lambda)^{-1/2}\mu_2'(0)\eta^{10}_{,\hat{t}} = 0 \quad (2.31e)$$

$$O(\epsilon^3\delta): \eta^{31}_{,\hat{t}}(0,0) + (1-\lambda)^{-1/2}\{\mu_2'(0)\eta^{11}_{,\hat{t}}(0,0) + \eta^{30}_{,\tau}(0,0)\} = 0 \quad (2.31f)$$

$$O(\epsilon^3\delta^2): \eta^{32}_{,\hat{t}}(0,0) + (1-\lambda)^{-1/2}\{\mu_2'(0)\eta^{12}_{,\hat{t}}(0,0) + \eta^{31}_{,\tau}(0,0)\} = 0 \quad (2.31g)$$

3.1Solution of the equations of orders η^{ij} *of the Model Problem*

From equation $(2.27a)$, *we have*

$$\eta^{10}_{,\hat{t}\hat{t}} + \eta^{10} = \frac{\lambda\cos(\omega)}{(1-\lambda\cos(\omega))}$$

with initial conditions $(2.31a)$*and* $(2.31b)$

Solving $(2.27a)$, results to,

$$\eta^{10}(\hat{t},\tau) = a_{10}(\tau)cos\hat{t} + b_{10}(\tau)sin\hat{t} + B(\tau) \qquad (3.1a)$$

where,

$$B(\tau) = \frac{\lambda\cos(\omega)}{(1-\lambda\cos(\omega))} \qquad (3.1b)$$

$\boldsymbol{Using the initial conditions (2.31a, b) in (2.32), gives}$

$$\eta^{10}(0,0) = a_{10}(0) + B(0) = 0$$

$\therefore a_{10}(0) = -B(0) = -\frac{\lambda}{1-\lambda}$

$From (2.31b), it\ follows\ that,$

$$\eta^{10}_{,\hat{t}}(0,0) = b_{10}(0) = 0$$

$\therefore b_{10}(0) = 0$

$Next, solving\ (2.27b), gives,$

$$\eta^{11}_{,\hat{t}\hat{t}} + \eta^{11} = \frac{-2}{(1-\lambda\cos(\omega))^{1/2}}\{\eta^{10}_{,\hat{t}\tau} + \eta^{10}_{,\hat{t}}\} - \frac{\lambda\omega'\sin(\omega)\,\eta^{10}_{,\hat{t}}}{2(1-\lambda\cos(\omega))^{3/2}} \qquad (3.2a)$$

$with\ initial\ conditions,$

$$\eta^{11}(0,0) = 0, \eta^{11}_{,\hat{t}}(0,0) + (1-\lambda)^{-1/2}\eta_{,\tau}(0,0) = 0.$$

$$Substituting\ for\ \eta^{10}_{,\hat{t}}\ and\ \eta^{10}_{,\hat{t}\tau}\ in\ (2.2a), gives,$$

$$\eta^{11}_{,\hat{t}\hat{t}} + \eta^{11} = \left\{\frac{2(a'_{10}+a_{10})}{(1-\lambda\cos(\omega))^{1/2}} + \frac{a_{10}\lambda\omega'\sin(\omega)}{2(1-\lambda\cos(\omega))^{3/2}}\right\}sin\hat{t} - \left\{\frac{2(b'_{10}+b_{10})}{(1-\lambda\cos(\omega))^{1/2}} + \frac{2(b'_{10}+b_{10})}{(1-\lambda\cos(\omega))^{1/2}}\right\}cos\hat{t} \qquad (3.2b)$$

To ensure a uniformly valid asymptotic solution in $\hat{t}$, it is necessary to equate to zero the coefficients of $sin\hat{t}\ and\ cost\ \hat{t}$ respectively in (3.2b).

Thus, it follows from the coefficients of $sin\hat{t}$, that,

$$\frac{2(a'_{10}+a_{10})}{(1-\lambda\cos(\omega))^{1/2}} + \frac{a_{10}\lambda\omega'\sin(\omega)}{2(1-\lambda\cos(\omega))^{3/2}} = 0 \qquad (3.2c)$$

$and\ from\ the\ coefficient\ of\ cos\hat{t}, the\ result\ is$

$$\frac{2(b'_{10}+b_{10})}{(1-\lambda\cos(\omega))^{1/2}} + \frac{b_{10}\lambda\omega'\sin(\omega)}{2(1-\lambda\cos(w))^{3/2}} = 0 \qquad (3.2d)$$

Solving (3.2c), gives,

$$\frac{a'_{10}}{a_{10}} + \frac{\lambda\omega'\sin(\omega)}{4(1-\lambda\cos(\omega))} = -1 \qquad (3.2e)$$

$Now\ integrating\ both\ sides (4.23e)\ with\ respect\ to\ \tau, gives$

$$Ina_{10} + {}^{1}/_{4}\,In(1-\lambda\cos(\omega)) = -\tau + C_1$$

where C_1 is an arbitrary constant.

$i.e \quad Ina_{10}(1-\lambda\cos(\omega))^{1/4} = -\tau + C_1, \therefore a_{10}(1-\lambda\cos(\omega))^{1/4} = e^{-\tau+c_1}$

$a_{10}(1-\lambda\cos(\omega))^{1/4} = A_o e^{-\tau}, \therefore a_{10}(\tau) = \frac{A_0 e^{-\tau}}{(1-\lambda\cos(\omega))^{1/4}}, where\ A_o = e^{c_1}$(constant)

$a_{10}(0)(1-\lambda)^{1/4} = A_0 = -B(0)(1-\lambda)^{1/4}, \therefore a_{10}(\tau) = -B(0)e^{-\tau}\left[\frac{1-\lambda}{(1-\lambda\cos(\omega))}\right]^{1/4}$

Similarly, solving (3.2d), gives,

$$\frac{b'_{10}}{b_{10}} + \frac{\lambda\omega'\sin(\omega)}{4(1-\lambda\cos(\omega))} = -1 \qquad (3.2f)$$

$Integrating\ both\ sides\ of\ (3.2f) w.r.t\ \tau, gives,$

$$Inb_{10} + \frac{1}{4}In(1-\lambda\cos(\omega)) = -\tau + C_2$$

where C_2 is an arbitrary constant.

$i.e \quad Inb_{10}(1-\lambda\cos(\omega))^{1/4} = -\tau + C_2$ and $b_{10}(\tau)(1-\lambda\cos(\omega))^{1/4} = e^{-\tau+c_2} = A_1 e^{-\tau}$

$where\ A_1 = e^{c_2}$ and for $\tau = 0$

$b_{10}(0)(1-\lambda)^{1/4} = A_1 = 0\ and\ b_{10}(\tau) = 0$

Substituting $b_{10}(\tau)$ *in* (3.1), *gives*

$$\eta^{10}(\hat{t},\tau) = a_{10}(\tau)\cos\hat{t} + B(\tau) \qquad (3.3)$$

Furthermore, solving the remaining part of equation $(3.2b)$ *gives*

$$\eta^{11}(\hat{t},\tau) = a_{11}(\tau)cost\,\hat{t} + b_{10}(\tau)sin\hat{t} \qquad (3.4a)$$

Then, applying the initial conditions in $(2.31b)$, *gives*

$\eta^{11}(0,0) = a_{11}(0) = 0, \therefore a_{11}(0) = 0$

From $\eta,_{\hat{t}}^{11}(0,0) = b_{11}(0) = 0$, thus, $b_{11}(0) = 0$

Now solving $(2.27c)$, *gives*

$$\eta,_{\hat{t}\hat{t}}^{12} + \eta^{12} = \frac{-2}{(1-\lambda\cos(\omega))^{1/2}}\{\eta,_{\hat{t}\tau}^{11} + \eta,_{\hat{t}}^{11}\} - \frac{1}{(1-\lambda\cos(\omega))^{1/2}}\{\eta,_{\tau\tau}^{10} + 2\eta_{\tau}^{10}\} - \frac{\lambda\omega'\sin(\omega)\,\eta,_{\hat{t}}^{11}}{2(1-\lambda\cos(\omega))^{3/2}} \qquad (3.4b)$$

with the initial conditions

$$\eta^{12}(0,0) = 0 \text{ and } \eta,_{\hat{t}}^{12}(0,0) + (1-\lambda)^{-1/2}\,\eta,_{\tau}^{11}\,\tau(0,0) = 0 \qquad (3.4c)$$

Substituting $\eta,_{\hat{t}}^{11}, \eta,_{\hat{t}\tau}^{11}, \eta,_{\tau}^{10}$ *and* $\eta,_{\tau\tau}^{10}$ *in* $(2.27c)$, *gives*

$$\eta_{,\hat{t}\hat{t}}^{12} + \eta^{12} = \left\{\frac{2(a_{11}' + a_{11})}{(1-\lambda\cos(\omega))^{1/2}} + \frac{a_{11}\lambda\omega'\sin(\omega)}{2(1-\lambda\cos(\omega))^{3/2}} - \frac{\left(b_{10}'' + 2b_{10}'\right)}{(1-\lambda\cos(\omega))}\right\} sin\hat{t}$$

$$-\left\{2\left(\frac{b_{11}' + b_{11}}{(1-\lambda\cos(\omega))^{1/2}} + \frac{b_{11}\lambda\omega'\sin(\omega)}{2(1-\lambda\cos(\omega))^{3/2}} - \frac{\left(a_{10}'' + 2a_{10}'\right)}{(1-\lambda\cos(\omega))}\right)\right\} cos\hat{t} - \frac{\left(B''(\tau) + 2B'(\tau)\right)}{(1-\lambda\cos(\omega))} \qquad (3.5a)$$

To ensure a uniformly valid asymptotic solution in $\hat{t}$, *there is need to equate to zero the coefficients of* $sin\hat{t}$ *and* $cost\,\hat{t}$ *respectively in* $(3.5a)$.

The coefficient of $sin\hat{t}$, *gives*

$$\frac{2(a_{11}' + a_{11})}{(1-\lambda\cos(\omega))^{1/2}} + \frac{a_{11}\lambda\omega'\sin(\omega)}{2(1-\lambda\cos(\omega))^{3/2}} - \frac{\left(b_{10}'' + 2b_{1o}'\right)}{(1-\lambda\cos(\omega))} = 0 \qquad (3.5b)$$

From the coefficient of $cos\hat{t}$, *the simplification gives*

$$\frac{2(b_{11}' + b_{11})}{(1-\lambda\cos(\omega))^{1/2}} + \frac{b_{11}\lambda\omega^{1}\sin(\omega)}{2(1-\lambda\cos(\omega))^{3/2}} - \frac{\left(a_{10}'' + 2a_{10}'\right)}{(1-\lambda\cos(\omega))} = 0 \qquad (3.5c)$$

Solving equation $(3.5b)$, *gives*

$$\frac{2(a_{11}' + a_{11})}{(1-\lambda\cos(\omega))^{1/2}} + \frac{a_{11}\lambda\omega'\sin(\omega)}{2(1-\lambda\cos(\omega))^{3/2}} = 0 \qquad (3.5d)$$

Because $b_{10}(\tau) = 0$, *then*, $b_{10}' = b_{10}'' = 0$

$$i.e \quad \frac{a_{11}'}{a_{11}} + \frac{\lambda\omega'\sin(\omega)}{4(1-\lambda\cos(\omega))} = -1 \qquad (3.5e)$$

Integrating both sides of $(3.5d)\omega.r.t.\tau$, *gives*

$$Ina_{11} + \frac{1}{4}In(1-\lambda\cos(\omega)) = -\tau + C_3$$

Where C_3 is an arbitrary constant.

$$i.e \quad Ina_{11}(1-\lambda\cos(\omega))^{1/4} = -\tau + C_3$$

$a_{11}(\tau)(1-\lambda\cos(\omega))^{1/4} = e^{-\tau+C_3} = A_3e^{-\tau}(where\ A_3 = e^{C_3})$

$a_{11}(0)(1-\lambda)^{1/4} = A_3 = 0, \therefore a_{11}(\tau) = 0$

Similarly, solving $(3.5c)$, *gives*

$$\frac{2(b_{11}' + b_{11})}{(1-\lambda\cos(\omega))^{1/2}} + \frac{b_{11}\lambda\omega'\sin(\omega)}{2(1-\lambda\cos(\omega))^{3/2}} = \frac{\left(a_{10}'' + 2a_{10}\right)}{(1-\lambda\cos(\omega))} \qquad (3.5f)$$

$$i.e\ b_{11}' + b_{11}\left\{1 + \frac{\lambda\omega'\sin(\omega)}{4(1-\lambda\cos(\omega))}\right\} = \frac{\left(a_{10}'' + 2a_{10}\right)}{2(1-\lambda\cos(\omega))} \qquad (3.5g)$$

$Integrating\ both\ sides\ of\ (3.5g)\ gives,$

$$b_{11}(\tau)\, e^{\tau}(1-\lambda\cos(\omega))^{1/4} = \left[\int_0^{\tau} \frac{e^{s}(a''_{10}+2a'_{10})ds}{2(1-\lambda\cos(\omega))^{1/4}} + C_4\right] \quad (3.5I)$$

$b_{11}(0) = (1-\lambda)^{-1/4}[0+C_4] = C_4(1-\lambda)^{-1/4} = 0, \tau = 0, \therefore C_4 = 0$

$Hence,$

$$b_{11}(\tau) = e^{-\tau}(1-\lambda\cos(\omega))^{-1/4}\left[\int_0^{\tau} \frac{e^{s}\left(a''_{10}+2a'_{10}\right)ds}{2(1-\lambda\cos(\omega))^{1/2}}\right]$$

$$\therefore \eta^{11}(\hat{t},\tau) = b_{11}\ (\tau)sin\hat{t};(because\ a_{11}\ (\tau) = 0\ ) \quad (3.6)$$

There remaining part of equation (3.5a) is

$$\eta^{12}_{,\hat{t}\hat{t}} + \eta^{12} = -\left(\frac{B''(\tau)+2B'(\tau)}{(1-\lambda cos(\omega))}\right) \quad (3.7a)$$

Solving (3.7a), the result gives

$$\eta^{12}(t,\tau) = a_{12}(\tau)cos\hat{t} + b_{12}(\tau)sin\hat{t} - \left(\frac{B''(\tau)+2B'(\tau)}{(1-\lambda\cos(\omega))}\right) \quad (3.7b)$$

Applying the initial conditions (2.31a) and (2.31d) in (3.7b), gives

$$\eta^{12}(0,0) = a_{12}(0) - \frac{B''(0)}{(1-\lambda)} = 0\ ; becuase B'(0) = 0$$

$$\therefore a_{12}(0) = \frac{B''(0)}{(1-\lambda)} = \frac{B(0)\omega'^2(0)(2\lambda-1)}{(1-\lambda)^2}\ and\ b_{12}(0) = 0 \quad (3.7c)$$

3.2 Solution of the equations of orders $\boldsymbol{\eta^{3j}, j = 0,1,2\ of\ the\ model\ problem}$

The next solution contains terms such as $(\eta^{10})^3$ which need to be evaluated before substitution. Doing this, the following is obtained

$$(\eta^{10})^3 = (a_{10}cos\hat{t}+B)^3 = \left(B^3 + \frac{3a_{10}^2B}{2}\right) + \left(\frac{3a_{10}^3}{4} + 3a_{10}B^2\right)cos\hat{t} + \frac{3}{2}a_{10}^2cos2\hat{t} + \frac{a_{10}^3}{4}cos3\hat{t} \quad (3.8)$$

Substituting (3.8) in (2.28d), gives

$$\eta^{30}_{\hat{t}\hat{t}} + \eta^{30} = \frac{3b}{(1-\lambda\cos(\omega))}\left[\left(B^3+\frac{3a_{10}^2B}{2}\right) + \left(\frac{3a_{10}^3}{4}+3a_{10}B^2\right)cos\hat{t} + \frac{3}{2}a_{10}^2cos2\hat{t} + \frac{a_{10}^3}{4}cos3\hat{t}\right] + \frac{2\mu'_2a_{10}cos\hat{t}}{(1-\lambda\cos(\omega))^{1/2}} \quad (3.9)$$

$with\ initial\ conditions,$

$$\eta^{30}(0,0) = 0;\ \eta^{30}_{,\hat{t}}(0,0) + (1-\lambda)^{1/2}\left[\mu'_2(0)\eta^{10}_{,\hat{t}}(0,0)\right] = 0$$

$Ensuring\ a\ uniformly\ valid\ asymptotic\ solution\ in\ \hat{t}, needs\ equating\ to\ zero\ the\ coefficients\ of\ \sin\hat{t}\ and\ cos\hat{t}\ respectively\ in\ (3.9).$

$Thus, the\ coefficient\ of\ cos\hat{t}, gives$

$$\frac{3b}{(1-\lambda\cos(\omega))}\left\{\frac{a_{10}^3}{4} + a_{10}B^2\right\} + \frac{2\mu'_2a_{10}}{(1-\lambda\cos(\omega))^{1/2}} = 0 \quad (3.10)$$

$From\ (3.10), it\ follows\ that$

$$\mu'_2 = \frac{-3b(a_{10}^2+4B^2)}{8(1-\lambda\cos(\omega))^{1/2}} \quad (3.11)$$

$The\ remaining\ equation\ (3.9)\ is$

$$\eta^{30}_{,\hat{t}\hat{t}} + \eta^{30} = \frac{b}{(1-\lambda\cos(\omega))}\left[\left(\frac{3}{2}a_{10}^2B + B^3\right) + \frac{3}{2}a_{10}^2Bcos2\hat{t} + \frac{3}{4}a_{10}^3cos3\hat{t}\right] \quad (3.12a)$$

The solution (3.12*a*) *is*

$$\eta^{30}(\hat{t},\tau) = a_{30}(\tau)cos\hat{t} + b_{30}(\tau)sin\hat{t} + \frac{b}{(1-\lambda\cos(\omega))}\left[\left(\frac{3a_{10}^2B}{2}+B^3\right)-\frac{Ba_{10}^2}{2}cos2\hat{t} - \frac{3a_{10}^3cos3\hat{t}}{32}\right] \quad (3.12b)$$

$$\eta^{30}(\hat{t},\tau) = a_{30}(\tau)cos\hat{t} + b_{30}(\tau)sin\hat{t} + b\left[S_0(\tau) + S_1(\tau)cos2\hat{t} + S_2(\tau)cos3\hat{t}\right] \quad (3.12c)$$

where,

$$S_0(\tau) = \frac{1}{(1-\lambda\cos(\omega))}\left(\frac{3}{2}Ba_{10}^2 + B^3\right) \quad (3.12d)$$

$$S_1(\tau) = \frac{-Ba_{10}^2}{2(1-\lambda\cos(\omega))}\ ;\ S_2(\tau) = \frac{-3a_{10}^3}{32(1-\lambda\cos(\omega))} \quad (3.12e)$$

and where,

$$S_0(0) = \frac{5B^3(0)}{2(1-\lambda)}\ ; S_1(0) = \frac{-B^3(0)}{2(1-\lambda)}\ ; S_2(0) = \frac{3B^3(0)}{32(1-\lambda)} \quad (3.12f)$$

Using the initial conditions in (3.12*c*), *gives*

$$\eta^{30}(0,0) = a_{30}(0) + b[S_0(0) + S_1(0) + S_2(0)] = 0$$

$$\therefore a_{30}(0) = -b\left[\frac{5B^3(0)}{2(1-\lambda)} - \frac{B^3(0)}{2(1-\lambda)} + \frac{3B^3(0)}{32(1-\lambda)}\right] = -\frac{67\ bB^3(0)}{32(1-\lambda)};\ b_{30}(0) = 0 \quad (3.12g)$$

In the next solution, there shall be the need for terms like $B'(0), S_0'(0), S_1'(0)$ *and* $S_2'(0)$ *which are neccessary and are evaluated as follows*

$$B'(\tau) = (1-\lambda\cos(\omega))\left[\frac{-\lambda w'\sin(\omega) - \lambda^2\omega'\sin(\omega)\cos(\omega)}{(1-\lambda\cos(\omega))^2}\right];\ \therefore B'(0) = 0$$

$$S_0'(\tau) = \frac{1}{(1-\lambda\cos(\omega))}\left[\frac{3}{2}(2Ba_{10}a_{10}' + B'a_{10}^2 + 3B^2B) + \left(\frac{3Ba_{10}^2}{2}+B^3\right)\left(\frac{-\lambda\omega'\sin(\omega)}{(1-\lambda\cos(\omega))^2}\right)\right], S_0'(0) = 0$$

$$S_1'(\tau) = -\frac{1}{2}\left[\left(\frac{B'a_{10}^2 + 2Ba_{10}a_{10}'}{(1-\lambda\cos(\omega))}\right) + \left(\frac{Ba_{10}^2\lambda\omega'\sin(\omega)}{(1-\lambda\cos(\omega))^2}\right)\right], S_1'(0) = 0$$

$$S_2'(\tau) = \frac{-3}{32}\left[\frac{3a_{10}'a_{10}^2}{(1-\lambda\cos(\omega))} - \frac{a_{10}^3\lambda\omega'\sin(\omega)}{(1-\lambda\cos(\omega))^2}\right],\quad S_2'(0) = 0$$

The following equation is obtained on substituting in (2.27*e*)

$$\eta,_{\hat{t}\hat{t}}^{31} + \eta^{31} = \frac{3b(\eta^{10})^2\eta^{11}}{(1-\lambda\cos(\omega))} - \frac{2\mu_2'\eta,_{\hat{t}\hat{t}}^{11}}{(1-\lambda\cos(\omega))^{1/2}} - \frac{2\eta,_{\hat{t}}^{30}}{(1-\lambda\cos(\omega))^{1/2}} - \frac{2\eta,_{\hat{t}\tau}^{30}}{(1-\lambda\cos(\omega))^{1/2}} - \frac{2\mu_2'\eta,_{\hat{t}\tau}^{10}}{(1-\lambda\cos(\omega))}$$
$$- \frac{\mu_2''\eta,_{\hat{t}}^{10}}{(1-\lambda\cos(\omega))} - \frac{2\mu_2'\eta^{10}}{(1-\lambda\cos(\omega))} - \frac{\lambda\omega'\sin(\omega)\eta,_{\hat{t}}^{30}}{2(1-\lambda\cos(\omega))^{3/2}} \quad (3.13)$$

with initial conditions,

$$\eta^{31}(0,0) = 0\ and \eta,_{\hat{t}}^{31}(0,0) + (1-\lambda)^{1/2}\left[\mu_2'(0)\eta,_{\hat{t}}^{11}(0,0) + \eta,_{\tau}^{30}(0,0)\right] = 0$$

Next, it is neccessary to evaluate the term $(\eta^{10})^2\eta^{11}$ *first, before going forward*

The following is obtained in this case,

$$(\eta^{10})^2\eta^{11} = b_{11}\left(\frac{a_{10}^2}{4} + B\right)sin\hat{t} + b_{11}Ba_{10}sin2\hat{t} + \frac{b_{11}a_{10}^2sin3\hat{t}}{4} \quad (3.14)$$

Substituting (3.14) *in* (3.13), *gives*

$$\eta_{\hat{t}\hat{t}}^{31} + \eta^{31} = \frac{3b}{(1-\lambda\cos(\omega))}\left[b_{11}\left(\frac{a_{10}^2}{4} + B^2\right)sin\hat{t} + b_{11}Ba_{10}sin2\hat{t} + \frac{b_{11}a_{10}^2sin3\hat{t}}{4}\right] + \frac{2\mu_2'b_{11}sin\hat{t}}{(1-\lambda\cos(\omega))^{1/2}}$$
$$\frac{-2}{(1-\lambda\cos(\omega))^{1/2}}\left\{(-a_{30}sin\hat{t} + b_{30}\widehat{cos\hat{t}}) + b\,(-2S_1sin2\hat{t} - 3S_2sin3\hat{t})\right\}$$

$$\frac{-2}{(1-\lambda\cos(\omega))^{1/2}}[-a'_{30}sin\hat{t}+b'_{30}cos\hat{t}+b(-2S'_1 sin2\hat{t}-3S'_2 sin3\hat{t})]+\frac{2\mu'_2 a^1_{10}sin\hat{t}}{(1-\lambda\cos(\omega))}$$
$$+\frac{\mu''_2 a_{10}sin\hat{t}}{(1-\lambda\cos(\omega))}+\frac{2\mu'_2\, a_{10}sin\hat{t}}{(1-\lambda\cos(\omega))}-\frac{\lambda\omega'\sin(\omega)}{2(1-\lambda\cos(\omega))^{3/2}}\{-a_{30}sin\hat{t}+b_{30}cos\hat{t}$$
$$+b(-2S_1 sin2\hat{t}-3S_2 sin3\hat{t}\} \qquad (3.15)$$

To ensure a uniformly valid asymptotic solution in $\hat{t}$, *it neccessary to equate to zero the coefficients of* $\sin\hat{t}$ *and* $cos\hat{t}$ *respectively in* (3.15).

From the coefficient of $sin\hat{t}$, *the resultant equation is*

$$\frac{3bb_{11}}{4(1-\lambda\cos(\omega))}(a^2_{10}+4B^2)+\frac{2\mu'_2 b_{11}}{(1-\lambda\cos(\omega))^{1/2}}+\frac{2a_{30}}{(1-\lambda\cos(\omega))^{1/2}}+\frac{2a'_{30}}{(1-\lambda\cos(\omega))^{1/2}}$$
$$+\frac{2\mu'_2 a'_{10}}{(1-\lambda\cos(\omega))}+\frac{\mu''_2 a_{10}}{(1-\lambda\cos(\omega))}+\frac{2\mu'_2\ a_{10}}{(1-\lambda\cos(\omega))}+\frac{a_{30}\lambda\omega'\sin(\omega)}{2(1-\lambda\cos(\omega))^{3/2}}=0 \qquad (3.16a)$$

For the coefficient of $cos\hat{t}$, *it easily follows that*

$$\frac{2b'_{30}}{(1-\lambda\cos(\omega))^{1/2}}+\frac{2b_{30}}{(1-\lambda\cos(\omega))^{1/2}}+\frac{b_{30}\lambda\omega'\sin(\omega)}{2(1-\lambda\cos(\omega))^{3/2}}=0 \qquad (3.16b)$$

A rearrangement of (3.16*a*) *gives*

$$\frac{2a'_{30}}{(1-\lambda\cos(\omega))^{1/2}}+\frac{2a_{30}}{(1-\lambda\cos(\omega))^{1/2}}+\frac{a_{30}\lambda\omega'\sin(\omega)}{2(1-\lambda\cos(\omega))^{3/2}}$$
$$=-\left[\frac{3bb_{11}}{4(1-\lambda\cos(\omega))}(a^2_{10}+B^2)+\frac{2\mu'_2 b_{11}}{(1-\lambda\cos(\omega))^{1/2}}\right.$$
$$\left.+\frac{2\mu'_2 a'_{10}}{(1-\lambda\cos(\omega))}+\frac{2\mu'_2 a_{10}}{(1-\lambda\cos(\omega))}+\frac{\mu''_2 a_{10}}{(1-\lambda\cos(\omega))}\right]$$

$$i.e\quad a'_{30}+\ a_{30}\left(1+\frac{\lambda\omega'\sin(\omega)}{4(1-\lambda\cos(\omega))}\right)=q_1\quad(\tau) \qquad (3.16c)$$

where,

$$q_1\quad(\tau)=-\left(\frac{1-\lambda\cos(\omega)}{2}\right)^{1/2}\left[\frac{3bb_{11}}{(1-\lambda\lambda\cos(\omega))^{1/2}}(a^2_{10}+B^2)+\frac{2\mu'_2 b_{11}}{(1-\lambda\lambda\cos(\omega))^{1/2}}+\frac{2\mu'_2 a'_{10}}{(1-\lambda\cos(\omega))}+\frac{\mu''_2 a_{10}}{(1-\lambda\cos(\omega))}+\frac{2\mu'_2 a_{10}}{(1-\lambda\cos(\omega))^{1/2}}\right] \text{(3.16d)}$$

Solving (3.16c) gives,

$$\therefore a_{30}(\tau)=e^{-\tau}(1-\lambda\cos(\omega))^{-1/4}\left[\int_0^\tau q_1(s)e^s(1-\lambda cos\omega)^{1/4}+C_5\right]$$

$$a_{30}(0)=(1-\lambda)^{-1/4}[0+C_5];\ C_5=\frac{67bB^3(0)}{32(1-\lambda)^{3/4}};\left(because\ \ a_{30}(0)=-\frac{67bB^3(0)}{32(1-\lambda)}\right)$$

Similarly, solving (3.16*b*) *gives,*

$$b'_{30}+b_{30}\left(1+\frac{\lambda\omega'\sin(\omega)}{4(1-\lambda\cos(\omega))}\right)=0 \qquad (3.16f)$$

Solving (3.16f) gives,

$$b_{30}(1-\lambda\cos(\omega))^{1/4}e^{\,\tau}=C_6;\ \therefore b_{30}(\tau)=C_6(1-\lambda\cos(\omega))^{-1/4}\,e^{-\tau};b_{30}(0)=0;\therefore C_6=0$$

Hence,

$$b_{30}(\tau)=0$$

Substituting $b_{30}(\tau)$ *in* (3.12*c*) , *gives*

$$\eta^{30}(\hat{t},\tau)=a_{30}(\tau)cos\hat{t}+\ b\,[S_0(\tau)+S_1(\tau)cos2\hat{t}+S_2(\tau)cos3\hat{t}] \qquad (3.16h)$$

The remaining part of equation (3.15) *is solved to get*

$$\eta^{31}(\hat{t},\tau)=a_{31}(\tau)cos\hat{t}+b_{31}(\tau)sin\hat{t}-\frac{S_3}{3}sin2\hat{t}-\frac{S_4 sin3\hat{t}}{8} \qquad (3.16i)$$

where,

$$S_3(\tau) = \frac{3bb_{11}Ba_{10}}{(1-\lambda\cos(\omega))} + \frac{bS_1\lambda\omega'\sin(\omega)}{2(1-\lambda\cos(\omega))^{3/2}} + \frac{4bS_1}{(1-\lambda cos\omega)^{1/2}} + \frac{4bS_1'}{(1-\lambda cos\omega)^{1/2}} \quad (3.17a)$$

$$S_4(\tau) = \frac{3bb_{11}a_{10}^2}{4(1-\lambda\cos(\omega))} + \frac{3bS_2\lambda\omega'\sin(\omega)}{2(1-\lambda\cos(\omega))^{3/2}} + \frac{6bS_2}{(1-\lambda cos\omega)^{1/2}} + \frac{6bS_2'}{(1-\lambda cos\omega)^{1/2}} \quad (3.17b)$$

$$S_3(0) = \frac{20bB^3(0)}{(1-\lambda)^{1/2}} = bB^3(0)S_5; S_5 = \frac{20}{(1-\lambda)^{1/2}};\ S_4(0) = \frac{-18bB^3(0)}{32(1-\lambda)^{1/2}} = bB^3(0)S_6; S_6 = \frac{-18}{32(1-\lambda)^{1/2}}$$

Using the initial condition, $\eta^{31}(0,0) = 0$ *in* $(3.16h)$*, gives,*

$$\eta^{31}(0,0) = a_{31}(0) = 0;\ \therefore a_{31}(0) = 0$$

Using the second initial condition $(2.31f)$ *in* $(3.16h)$*, gives*

$$b_{31}(0) - \frac{2}{3}S_2(0) - \frac{3}{8}S_3(0) + (1-\lambda)^{-1/2}\left[a_{30}(0) + b\left(S_0'(0) + S_1'(0) + S_2'(0)\right)\right] = 0$$

$\therefore b_{31}(0) = 0;$ *(because* $S_0'(0) = S_1'(0) = S_2'(0) = 0$*)*

Solving the equation $(3.27f)$ *gives*

$$\eta,^{32}_{\hat{t}\hat{t}} + \eta^{32} = \frac{3b}{(1-\lambda\cos(\omega))}[\eta^{10}(\eta^{11})^2 + (\eta^{10})^2\eta^{12}] - \frac{\lambda\omega'\sin(\omega)\,\eta,^{31}_{\hat{t}}}{2(1-\lambda\cos(\omega))^{3/2}} - \frac{2\eta,^{31}_{\hat{t}\tau}}{(1-\lambda\cos(\omega))^{1/2}}$$
$$- \frac{2\eta,^{31}_{\hat{t}}}{(1-\lambda\cos(\omega))^{1/2}} - \frac{2\mu_2'\eta,^{12}_{\hat{t}\hat{t}}}{(1-\lambda\cos(\omega))^{1/2}} - \frac{2\mu_2'\eta,^{11}_{\hat{t}\tau}}{(1-\lambda\cos(\omega))^{1/2}} - \frac{2\mu_2'\eta,^{11}_{\hat{t}}}{(1-\lambda\cos(\omega))} - \frac{\eta,^{30}_{\tau\tau}}{(1-\lambda\cos(\omega))}$$
$$- \frac{2\eta,^{30}_{\tau}}{(1-\lambda\cos(\omega))} - \frac{2\mu_2''\eta,^{11}_{\hat{t}}}{(1-\lambda\cos(\omega))} \quad (3.18a)$$

with the initial conditions,

$$\eta^{32}(0,0) = 0\ and\ \eta,^{32}_{\hat{t}}(0,0) + (1-\lambda)^{-1/2}\left[\mu_2'(0)\eta,^{12}_{\hat{t}}(0,0) + \eta,^{31}_{\tau}(0,0)\right] = 0$$

We need to evluate $\eta^{10}(\eta^{11})^2 + (\eta^{10})^2\eta^{12}$ *first before proceeding and obtain*

$$\eta^{10}(\eta^{11})^2 + (\eta^{10})^2\eta^{12} = \left[2Ba_{10}a_{12} - \frac{B''}{(1-\lambda cos\omega)}\left(\frac{a_{10}^2}{2} + B^2\right) + \frac{Bb_{11}^2}{2}\right] + \left[a_{12}\left(\frac{a_{10}^2 + 2B^2}{2}\right)\right.$$
$$\left. - \frac{2BB''a_{10}^2}{(1-\lambda cos\omega)} + \frac{a_{10}^2}{4} + \frac{b_{11}^2a_{10}}{4}\right]cos\hat{t} + \left[2Ba_{12}a_{10} - \frac{B''a_{10}^2}{(1-\lambda cos\omega)} - \frac{Bb_{11}^2}{2}\right]cos2\hat{t}$$
$$+ \left[\frac{a_{12}a_{10}^2}{4} - \frac{b_{10}^2a_{10}}{4}\right]cos\hat{t} \quad (3.18b)$$

Substituting $(3.18b)$ *in* $(3.18a)$ *gives*

$$\eta,^{32}_{\hat{t}\hat{t}} + \eta^{32} = \frac{3b}{(1-\lambda cos\omega)}\left[\left[\left\{\left(2Ba_{10}a_{12} - \frac{B''}{(1-\lambda\cos(\omega))}\right)\left(\frac{a_{10}^2 + 2B^2}{2}\right) + Bb_{11}^2\right\} + \left\{a_{12}\left(\frac{a_{10}^2 + 2B^2}{2}\right)\right.\right.\right.$$
$$\left.\left. - \frac{2BB''a_{10}^2}{(1-\lambda\cos(\omega))} + \frac{a_{10}^2}{4} + \frac{b_{11}^2a_{10}}{4}\right\}cos\hat{t}\right] + \left\{\left\{2Ba_{12}a_{10} - \frac{B''a_{10}^2}{(1-\lambda\cos(\omega))} - \frac{Bb_{11}^2}{2}\right\}\right\}cos2\hat{t}$$
$$+ \left\{\left(\frac{a_{12}a_{10}^2}{4} - \frac{b_{11}^2a_{10}}{4}\right)\right\}cos\hat{t} - \frac{\lambda\omega'\sin(\omega)}{2(1-\lambda\cos(\omega))^{3/2}}\left\{-a_{31}sin\hat{t} + b_{31}cos\hat{t} - \frac{2}{3}S_3cos2\hat{t} - \frac{3}{8}S_4sin3\hat{t}\right\}$$
$$- \frac{2}{(1-\lambda\cos(\omega))^{1/2}}\{-a_{31}'sin\hat{t} + b_{31}'cos\hat{t} - 2S_3'cos2\hat{t} + 3S_4'cos3\hat{t}\} - \frac{2}{(1-\lambda\cos(\omega))^{1/2}}\{-a_{31}sin\hat{t}$$
$$+ b_{31}cos\hat{t} + 2S_3cos2\hat{t} + 3S_4cos3\hat{t}\} - \frac{2\mu_2'}{(1-\lambda\cos(\omega))^{1/2}}\{-a_{12}cos\hat{t} - b_{12}sin\hat{t}\} - \frac{2\mu_2'b_{11}'cos\hat{t}}{(1-\lambda\cos(\omega))^{1/2}}$$
$$- \frac{\mu_2''b_{11}cos\hat{t}}{(1-\lambda\cos(\omega))} - \frac{2}{(1-\lambda cos\omega)}\{a_{30}'sin\hat{t} + b_{30}'cos\hat{t} + b(S_0' + S_1'cos2\hat{t} + S_2'cos3\hat{t})\} \quad (3.19a)$$

Ensuring a uniformly valid asymptotic solution in $\hat{t}$ *is achieved by equating to zero the coefficients of* $cos\hat{t}$ *and* $sin\hat{t}$ *respectively in* $(3.19a)$.

From the coefficient of $\cos\hat{t}$, *the resultant equation is*

$$\frac{3b}{(1-\lambda\cos(\omega))}\left[a_{12}\left(\frac{a_{10}^2+2B^2}{2}\right)-\frac{2BB^{"}a_{10}^2}{(1-\lambda\cos(\omega))}+\frac{a_{10}^2}{4}+\frac{b_{11}^2a_{10}}{4}\right]-\frac{b_{31}\lambda\omega'\sin(\omega)}{(1-\lambda\lambda\cos(\omega))^{3/2}}-\frac{2b_{31}'}{(1-\lambda\cos(\omega))^{1/2}}$$
$$-\frac{2b_{31}}{(1-\lambda\cos(\omega))^{1/2}}+\frac{2\mu_2'a_{12}}{(1-\lambda\cos(\omega))^{1/2}}-\frac{2\mu_2'b_{11}'}{(1-\lambda\cos(\omega))^{1/2}}-\frac{2\mu_2'b_{11}}{(1-\lambda\cos(\omega))}-\frac{2b_{30}'}{(1-\lambda\cos(\omega))}$$
$$-\frac{\mu_2'b_{11}}{(1-\lambda\cos(\omega))}-\frac{a_{30}^{"}}{(1-\lambda\cos(\omega))}-\frac{b_{30}'}{(1-\lambda\cos(\omega))}-\frac{\mu_2'b_{11}}{(1-\lambda\cos(\omega))}=0 \qquad (3.19b)$$

From the coefficient of $\sin\hat{t}$, *it follows that*

$$\frac{a_{31}\lambda\omega'\sin(\omega)}{2(1-\lambda\cos(\omega))^{3/2}}+\frac{2a_{31}'}{(1-\lambda\cos(\omega))^{1/2}}+\frac{2a_{31}}{(1-\lambda\cos(\omega))^{1/2}}+\frac{2\mu_2'b_{12}}{(1-\lambda\cos(\omega))^{1/2}}-\frac{2a_{30}'}{(1-\lambda\cos(\omega))}=0 \quad (3.19c)$$

A re_arrangement of equation $(3.19b)$ *gives*

$$b_{31}'+b_{31}\left(1+\frac{\lambda\omega'\sin(\omega)}{4(1-\lambda\cos(\omega))}\right)=q_3(\tau) \qquad (4.19d)$$

where,

$$q_3(\tau)\left(\frac{1-\lambda\cos\omega}{2}\right)^{1/2}\left[\frac{3b}{(1-\lambda\cos(\omega))}\left\{a_{12}\left(\frac{a_{10}^2+2B^2}{2}\right)-\frac{2BB^{"}a_{10}^2}{(1-\lambda\cos(\omega))}+\frac{a_{10}^2}{4}+\frac{b_{11}^2a_{10}}{4}\right\}+\frac{2\mu_2'a_{12}}{(1-\lambda\cos(\omega))^{1/2}}\right.$$
$$-\frac{2\mu_2'b_{11}'}{(1-\lambda\cos(\omega))^{1/2}}-\frac{2\mu_2'b_{11}}{(1-\lambda\cos(\omega))}-\frac{a_{30}'}{(1-\lambda\cos\omega)}-\frac{2b_{30}'}{(1-\lambda\cos\omega)}$$
$$\left.-\frac{\mu_2^{"}b_{11}}{(1-\lambda\cos(\omega))}\right]$$

Solving (3.19d), gives,

$$b_{31}(\tau)e^{\tau}(1-\lambda\cos(\omega))^{1/4}=\left[\int_0^{\tau}q_3(s)\,e^s(1-\lambda\cos(\omega))^{1/4}ds+C_7\right]$$
$$\therefore b_{31}(\tau)=e^{-\tau}(1-\lambda\cos(\omega))^{-1/4}\left[\int_0^{\tau}\rho_3(s)\,e^s(1-\lambda\cos(\omega))^{1/4}ds+C_7\right]$$
$$b_{31}(0)=(1-\lambda)^{-1/4}[0+C_7]=0;\therefore C_7=0;(because\ b_{31}(0)=0)$$

Then, It follows that

$$b_{31}(\tau)=e^{-\tau}(1-\lambda\cos(\omega))^{-1/4}\left[\int_0^{\tau}\rho_3(s)\,e^s(1-\lambda\cos(\omega))^{1/4}ds\right] \qquad (3.19f)$$

Solving $(3.19c)$ *gives*

$$a_{31}'+a_{31}\left(1+\frac{\lambda\omega'\sin(\omega)}{4(1-\lambda\cos(\omega))}\right)=\rho_4(\tau) \qquad (3.20a)$$

where,

$$\rho_4(\tau)=\left(\frac{1-\lambda\cos(\omega)}{2}\right)^{\frac{1}{2}}\left[\frac{a_{30}'}{(1-\lambda\cos(\omega))}-\frac{2\mu_2'b_{12}}{(1-\lambda\cos(\omega))^{1/2}}\right]$$

Solving (3.20a), gives,

$$a_{30}(\tau)e^{\tau}(1-\lambda\cos(\omega))^{1/4}=\left[\int_0^{\tau}\rho_4(\tau)e^s(1-\lambda\cos(\omega))^{1/4}ds+C_8\right]$$
$$a_{31}(0)=(1-\lambda)^{-1/4}[0+C_8]=0\therefore C_8=0;(since\ a_{31}(0)=0)$$

Then, it follows that,

$$a_{31}(\tau)=e^{-\tau}(1-\lambda\cos(\omega))^{-1/4}\left[\int_0^{\tau}\rho_4(s)e^s(1-\lambda\cos(\omega))^{1/4}ds\right] \qquad (3.20b)$$

The remaining part of the equation $(3.20a)$ *is*

$$\eta_{,\hat{t}\hat{t}}^{32} + \eta^{32} = S_7(\tau) + S_8(\tau)cos2\hat{t} + S_9(\tau)sin2\hat{t} + S_{10}(\tau)cos3\hat{t} + S_{11}(\tau)\, sin3\hat{t} \qquad (3.21)$$

where;

$$S_7(\tau) = \frac{3b}{(1-\lambda cos\omega)}\left[2B\, a_{10}a_{12} - \frac{B^{"}}{(1-\lambda\cos(\omega))}\left(\frac{a_{10}^2 + 2B^2}{2}\right) + \frac{Bb_{11}^2}{2} - \frac{2bs_0'}{(1-\lambda cos\omega)}\right] \qquad (3.22a)$$

$$S_8(\tau) = \frac{3b}{(1-\lambda\cos(\omega))}\left[2Ba_{12}a_{10} - \frac{B^{"}a_{10}^2}{(1-\lambda\cos(\omega))} - \frac{Bb_{11}^2}{2}\right] - \frac{2S_3\lambda\omega'\sin(\omega)}{6(1-\lambda\cos(\omega))^{3/2}} + \frac{4S_3'}{(1-\lambda\cos(\omega))^{1/2}}$$
$$-\frac{4S_3}{(1-\lambda\cos(\omega))^{1/2}} - \frac{2S_1'}{(1-\lambda\cos(\omega))} \qquad (3.22b)$$

$$S_9\ (\tau) = \frac{2S_1'(\tau)}{(1-\lambda\cos(\omega))} \qquad (3.22c)$$

$$S_{10}(\tau) = \left(\frac{a_{10}^2a_{12} - b_{11}^2a_{10}}{4}\right) + \frac{6S_4'}{(1-\lambda\cos(\omega))^{1/2}} - \frac{6S_4}{(1-\lambda\cos(\omega))^{1/2}} - \frac{S_2'}{(1-\lambda\cos(\omega))} \qquad (3.22d)$$

$$S_{11}(\tau) = \frac{3S_2'}{(1-\lambda\cos(\omega))} (3.22e)$$

But,

$$S_7(0) = -3bB^2(0)S_{12};\ S_{12} = \left\{\frac{6\omega'^2(0) + B(0)(1-\lambda)}{2(1-\lambda)^2}\right\};\ \ S_8(0) = 3b\, B(0)S_{13} \qquad (3.23a,b)$$

where,

$$S_{13} = \left[\left\{\frac{-4B^{"}(0) - 2B(0)B^{"}(0) - B^2(0)(1-\lambda)}{2(1-\lambda)^2}\right\} + \frac{4bB^2(0)S_{12}}{(1-\lambda)^{1/2}} - \frac{4bB^2(0)S_5}{(1-\lambda)^{1/2}}\right]$$

$$S_9(0) = S_{11}(0) = 0,\ \ S_{10}(0) = b^2BS_{14}(3.23c)$$

$$where \qquad S_{14} = \left[\left\{\frac{B^{"}(0) + B(0)(1-\lambda)}{4(1-\lambda)}\right\} + \frac{6bB(0)S_{13}}{(1-\lambda)^{1/2}} - \frac{6bB(0)S_6}{(1-\lambda)^{1/2}}\right]$$

The solution of (3.21) *is*

$$\eta^{32}(\hat{t},\tau) = a_{32}(\tau)cos\hat{t} + b_{32}(\tau)sin\hat{t} + S_7\ (\tau) - \frac{S_8\ (\tau)}{3}cos\hat{t}$$
$$-\frac{S_9(\tau)sin2\hat{t}}{3} - \frac{S_{10}(\tau)}{8}cos3\hat{t} - \frac{S_{11}(\tau)sin3\hat{t}}{8} \qquad (3.24)$$

Using the initial conditions, (3.31*a*)*and* (3.31*g*) *in* (4.24), *gives*

$\eta^{32}(0,0) = a_{32}(0) + S_7(0) - \frac{S_8\ (0)}{3} - \frac{S_{10}(0)}{8} = 0;\ \therefore a_{32}(0) = 3bB(0)S_{15}$

where,

$$S_{15} = \left[\frac{-24B(0)S_{12} + 8S_{13} + B(0)S_{14}}{24}\right]$$

4.3 The Maximum Displacement of the Model Problem

The displacement of $\eta(\hat{t},\tau)$ can be written as

$$\eta(\hat{t},\tau) = \epsilon(\eta^{10} + \delta\eta^{11} + \delta^2\eta^{12} + \cdots) + \epsilon^2(\eta^{20} + \delta\eta^{21} + \delta^2\eta^{22} + \cdots)$$
$$+\epsilon^3(\eta^{30} + \delta\eta^{31} + \delta^2\eta^{32} + \cdots) + \ \ldots \qquad (3.25a)$$

$$But\ \eta^{2j} = 0, \qquad j = 0,1,2,$$

Therefore(3.25*a*)*yields*,

$$\eta(\hat{t},\tau) = \epsilon\ (\eta^{10} + \delta\eta^{11} + \delta^2\eta^{12} + \cdots) + \epsilon^3(\eta^{30} + \delta\eta^{31} + \delta^2\eta^{32} + \cdots) + \cdots \quad (3.25b)$$

The formula for determining the dynamic buckling load is given as

$$\frac{d\lambda}{d\eta_c} = 0 \qquad (3.26)$$

where η_c *is the maximum of* $\eta(\hat{t},\tau)$ *and* η_c*is defined as* $\eta(\hat{t}_c,\tau_c)$, *where* $\hat{t}_c$*and* τ_c *are the critical values of* $\hat{t}$ *and* τ *respectively at maxmium displacement. We however* *need to determine* η_c *first before invoking*(3.26). *It is to be recalled that*, in terms *of the original* $t-$ *variable, the condition for maximum is*

$$\frac{d\eta}{dt}=0$$

However, the condition for maximum displacement interms of $\hat{t}$ *and* τ *is given as*

$$\eta_{,\hat{t}}+\left(\frac{\mu_2'\epsilon^2+\mu_3'\epsilon^3+\cdots}{(1-\lambda cos(\omega))^{1/2}}\right)\eta_{,\hat{t}}+\frac{\delta\eta_{,\tau}}{(1-\lambda\cos(\omega))^{1/2}}=0 \qquad (3.27)$$

It is pertinent to assume the following asymptotic series

$$\hat{t}_c=(\hat{t}_0+\delta\hat{t}_{01}+\delta^2\hat{t}_{02}+\cdots)+\epsilon(\hat{t}_{10}+\delta\hat{t}_{11}+\cdots)+\epsilon^2(\hat{t}_{20}+\delta\hat{t}_{21}+\cdots)+\cdots \qquad (3.28)$$

From (2.19b), we get,

$$\tau_c=\delta t_c \qquad (3.29a)$$

where t_c *is the critical value of* t *at maximum displacement. Expansion of* t_c *asymptotically, gives*

$$t_c=(T_0+\delta\, T_{01}+\delta^2T_{02}+\cdots)+\epsilon(T_{10}+\delta\, T_{11}+\cdots)+\epsilon^2(T_{20}+\delta T_{21}+\cdots)+\cdots \qquad (3.29b)$$

$$\therefore \tau_c=\delta t_c \quad =\delta[(T_0+\delta\, T_{01}+\cdots)+\epsilon(T_{10}+\delta\, T_{11}+\cdots)+\epsilon^2(T_{20}+\delta T_{21}+\cdots)] \qquad (3.29c)$$

At critical points, equation (3.27) gives

$$\eta,\hat{t}(\hat{t}_{c,}\tau_c)+\left[\frac{\mu_2'(\tau_c)\epsilon^2+\mu_3'(\tau_c)\epsilon^3+\cdots}{(1-\lambda cos\omega(\tau_c))^{1/2}}\right]\eta,\hat{t}(\hat{t}_{c,}\tau_c)+\frac{\delta}{(1-\lambda cos\omega(\tau_c))^{1/2}}\,\eta,\tau\,(\hat{t}_{c,}\tau_c)=0 \qquad (3.30)$$

The next thing is to expand (3.30) *asymptoticallyusing using* (3.28), (3.29*b*, *c*) *and obtain*

$$\begin{aligned}
&\epsilon\big[\eta^{10}_{,\hat{t}}+\{(\delta\hat{t}_{01}+\delta^2\hat{t}_{02}+\cdots)+\epsilon(\hat{t}_{10}+\delta s\hat{t}_{11}+\delta^2\hat{t}_{12}+\cdots)+\epsilon^2(\hat{t}_{20}+\delta\hat{t}_{21}+\cdots)\}\eta^{10}_{,\hat{t}\hat{t}}\\
&+\frac{1}{2}\{(\delta\hat{t}_{01}+\delta^2\hat{t}_{02}+\cdots)+\epsilon(\hat{t}_{10}+\delta\hat{t}_{11}+\delta^2\hat{t}_{12}+\cdots)+\epsilon^2(\hat{t}_{20}+\delta\hat{t}_{21}+\delta^2\hat{t}_{22}+\cdots)\}^2\eta^{10}_{,\hat{t}\hat{t}\hat{t}}\\
&+\delta\{(T_0+\delta\, T_{01}+\cdots)+\epsilon(T_{10}+\delta\, T_{11}+\cdots)+\epsilon^2(T_{20}+\delta\, T_{21}+\cdots)\}\eta^{10}_{,\hat{t}\tau}\\
&+2\delta\{(T_0+\delta\, T_{01}+\cdots)+\epsilon(T_{10}+\delta\, T_{11}+\cdots)+\epsilon^2(T_{20}+\delta\, T_{21}+\cdots)\}\{\delta\hat{t}_{01}+\cdots\\
&+\epsilon(\hat{t}_{10}+\delta\hat{t}_{11}+\cdots)+\epsilon^2(\hat{t}_{20}+\delta\hat{t}_{21}+\cdots)\}\eta^{10}_{,\hat{t}\hat{t}\tau}+\delta^2\{T_0+\cdots+\epsilon(T_{10}+\cdots)\\
&+\epsilon^2(T_{20}+\cdots)\}^2\eta^{10}_{,\hat{t}\tau\tau}\big]+\epsilon\delta\big[\eta,^{11}_{\hat{t}}\{(\delta\hat{t}_{01}+\cdots+\epsilon(\hat{t}_{10}+\delta\hat{t}_{11}+\cdots))+\epsilon^2(\hat{t}_{20}+\delta\hat{t}_{21}+\cdots)\}\eta,^{11}_{\hat{t}\hat{t}}\\
&+\delta\{T_0+\cdots+\epsilon(T_{10}+\cdots+)+\epsilon^2(T_{20}+\cdots+)\}\eta^{10}_{,\hat{t}\tau}+\frac{1}{2}\{(\delta\hat{t}_{01}+\cdots)+\epsilon(\hat{t}_{10}+\delta\hat{t}_{11}+\cdots)\\
&+\epsilon^2(\hat{t}_{20}+\delta\hat{t}_{21}+\cdots)\}^2\eta^{11}_{,\hat{t}}+2\delta\{(T_0+\cdots)+\epsilon(T_{10}+\cdots)+\epsilon^2(T_{20}\ldots+)\}\{\delta\hat{t}_{01}+\cdots\\
&+\epsilon(\hat{t}_{10}+\delta\hat{t}_{11})+\epsilon^2(\hat{t}_{20}+\delta\hat{t}_{21}+\cdots)\}\eta^{11}_{,\hat{t}\hat{t}\tau}\big]+\epsilon\delta^2\big[\eta^{12}_{,\hat{t}}+\{(\delta\hat{t}_{01}+\delta^2\hat{t}_{02}+\cdots)\\
&+\epsilon(\hat{t}_{10}+\cdots)+\epsilon^2(\hat{t}_{20}+\cdots)\}\eta^{12}_{,\hat{t}\hat{t}}+\cdots\big]+\epsilon^3\big[\eta^{30}_{,\hat{t}}+\{(\delta\hat{t}_{01}+\delta^2\hat{t}_{02}+\cdots)\}\eta^{30}_{,\hat{t}\hat{t}}+\delta\{T_0+\delta T_{01}\\
&+\cdots\}\eta^{30}_{,\hat{t}\tau}+\{\delta\hat{t}_{01}+\cdots\}^2\eta^{30}_{,\hat{t}\hat{t}\tau}+\cdots\big]+\epsilon^3\delta^2\big[\eta^{30}_{,\hat{t}}+\cdots\big]+\epsilon^3\left[\frac{\mu_2'(0)\eta^{10}_{,\hat{t}}}{(1-\lambda)^{1/2}}+\frac{\mu_2'(0)}{(1-\lambda)^{1/2}}\{\delta\hat{t}_{01}\right.\\
&+\delta^2\hat{t}_{02}+\cdots\}\eta^{10}_{,\hat{t}\hat{t}}+\delta\left(\frac{\mu_2'(0)\eta^{10}_{\hat{t}}}{1-\lambda\cos(\omega)^{1/2}}\right)_{,\tau}\{T_0+\delta T_{01}+\cdots\}+\delta\left(\frac{\mu_2'(0)}{1-\lambda\cos(\omega)^{1/2}}\right)\{T_0+\delta T_{01}+\cdots\}\\
&\left.+\frac{1}{2}\left(\frac{\mu_2'(0)}{1-\lambda\cos(\omega)^{1/2}}\right)\{\delta\hat{t}_{01}+\delta^2\hat{t}_{02}+\cdots\}\eta^{10}_{,\hat{t}\hat{t}\hat{t}}+2\delta\left(\frac{\mu_2'(\tau)\eta^{10}_{\hat{t}\hat{t}}}{1-\lambda\cos(\omega)^{1/2}}\right)_{,\tau}\{T_0+\cdots\}\{\delta\hat{t}_{01}+\cdots\}\right]\\
&+\delta^2\left\{\left(\frac{\mu_2'(\tau)\eta^{10}_{,\hat{t}}}{1-\lambda\cos(\omega)^{1/2}}\right)_{,\tau\tau}(T_0+\cdots)^2\right\}+\cdots+\epsilon^2\delta\left[\frac{\mu_2'(0)\eta^{11}_{,\hat{t}}}{(1-\lambda)^{1/2}}+\frac{\mu_2'(0)}{(1-\lambda)^{1/2}}\{\delta\hat{t}_{01}+\cdots\}\eta^{11}_{,\hat{t}\hat{t}}\right.\\
&+\delta\left(\frac{\mu_2'(\tau)\eta^{11}_{\hat{t}}}{1-\lambda\cos(\omega)^{1/2}}\right)_{,\tau}\{T_0+\cdots\}+\frac{1}{2}\left(\frac{\mu_2'(\tau)}{1-\lambda\cos(\omega)^{1/2}}\right)\{\delta\hat{t}_{01}+\cdots\}\eta^{11}_{,\hat{t}\hat{t}\hat{t}}
\end{aligned}$$

$$+2\delta\left(\frac{\mu_2'(\tau)\eta_{\hat{t}\hat{t}}^{11}}{1-\lambda\cos(\omega)^{1/2}}\right)_{,\tau}(T_0+\cdots)\Bigg]+\epsilon^3\delta^2\left[\frac{\mu_2'(0)\eta_{\hat{t}}^{12}}{(1-\lambda)^{1/2}}\right]\qquad=0 \qquad (3.31)$$

The next step is to proceed by equating (3.31) to zero in orders of $(\epsilon^i\delta^j)$and obtain the following equations.

$$O(\epsilon): \eta_{,\hat{t}}^{10}=0;\ O(\epsilon\delta): \hat{t}_{01}\eta_{,\hat{t}\hat{t}}^{10}+T_0\eta_{,\hat{t}\tau}^{10}+\eta_{\hat{t}}^{11}+\frac{\eta,\tau}{(1-\lambda)^{1/2}}=0 \qquad (3.32a,b)$$

$$O(\epsilon\delta^2): \hat{t}_{02}\eta_{,\hat{t}\hat{t}}^{10}+T_0\eta_{,\hat{t}\tau}^{11}+\frac{\hat{t}_{02}}{2}\eta_{,\hat{t}\hat{t}\hat{t}}^{10}+\hat{t}_{01}\eta_{,\hat{t}\hat{t}}^{11}+\eta_{,\hat{t}}^{12}+\frac{\hat{t}_{01}\eta_{,\hat{t}\tau}^{10}}{(1-\lambda)^{1/2}}$$
$$+\frac{\eta_{,\tau}^{11}}{(1-\lambda)^{1/2}}+T_0\eta_{,\hat{t}\tau}^{10}+2T_0\hat{t}_{01}\eta_{,\hat{t}\hat{t}\tau}^{10}=0 \qquad (3.32c)$$

$$O(\epsilon^2): \hat{t}_{01}\eta_{,\hat{t}\hat{t}}^{10}\quad=0;\ O(\epsilon^3): \hat{t}_{20}\eta_{,\hat{t}\hat{t}}^{10}\quad+\eta_{,\hat{t}}^{30}+\frac{\mu_2'(0)\eta_{\hat{t}}^{10}}{(1-\lambda)^{1/2}}=0 \qquad (3.32d,e)$$

$$O(\epsilon^3\delta): \hat{t}_{21}\eta_{,\hat{t}\hat{t}}^{10}\quad+T_{20}\eta_{,\hat{t}\tau}^{10}+2(T_0\hat{t}_{20}+T_{10}\hat{t}_{10})\eta_{,\hat{t}\hat{t}\tau}^{10}+\hat{t}_{20}\eta_{,\hat{t}}^{11}+\hat{t}_{01}\eta_{,\hat{t}\hat{t}}^{30}+T_0\eta_{,\hat{t}\tau}^{30}+\eta_{,\hat{t}}^{31}$$
$$+\frac{\hat{t}_{01}\mu_2'(0)\eta_{,\hat{t}\hat{t}}^{10}}{(1-\lambda)^{1/2}}+T_0\left(\frac{\mu_2'(\tau)\eta_{,\hat{t}}^{10}}{(1-\lambda\cos(\omega))^{1/2}}\right)_{,\tau}+2\,T_0\hat{t}_{01}\left(\frac{\mu_2'(\tau)\eta_{,\hat{t}\hat{t}}^{10}}{(1-\lambda\cos(\omega))^{1/2}}\right)_{,\tau}+\frac{\mu_2'(0)\eta_{,\hat{t}}^{10}}{(1-\lambda)^{1/2}}$$
$$+\frac{\hat{t}_{20}\eta_{,\hat{t}\tau}^{10}}{(1-\lambda)^{1/2}}+\frac{\eta_{,\tau}^{30}}{(1-\lambda)^{1/2}}=0 \qquad (3.32f)$$

$$O(\epsilon^3\delta^2): \hat{t}_{22}\eta_{,\hat{t}\hat{t}}^{10}\quad+T_{21}\eta_{,\hat{t}\tau}^{10}+2\hat{t}_{21}T_0\eta_{,\hat{t}\hat{t}\tau}^{10}+T_{10}^2\eta_{\hat{t}\hat{t}\tau}^{10}+\hat{t}_{21}\eta_{,\hat{t}\hat{t}}^{11}\quad+T_{20}\eta_{,\hat{t}\tau}^{11}+\hat{t}_{20}\eta_{,\hat{t}}^{12}+\hat{t}_{02}\eta_{,\hat{t}\hat{t}}^{30}$$
$$+\hat{t}_{01}\eta_{,\hat{t}\hat{t}}^{31}+T_0\eta_{,\hat{t}\tau}^{31}+\eta_{,\hat{t}}^{32}+\frac{\mu_2'(0)\hat{t}_{02}\eta_{,\hat{t}\hat{t}}^{10}}{(1-\lambda)^{1/2}}+\ T_{01}\left(\frac{\mu_2'(\tau)\eta_{,\hat{t}}^{10}}{(1-\lambda\cos(\omega))^{1/2}}\right)_{,\tau}+\left(\frac{\mu_2'(\tau)}{(1-\lambda\cos(\omega))^{1/2}}\right)\frac{\hat{t}_{01}}{2}\eta_{,\hat{t}\hat{t}\hat{t}}^{10}$$
$$+2T_0\hat{t}_{01}\left(\frac{\mu_2'(\tau)\eta_{,\hat{t}\hat{t}}^{10}}{(1-\lambda\cos(\omega))^{1/2}}\right)_{,\tau}+T_0^2\left(\frac{\mu_2'(\tau)\eta_{,\hat{t}}^{10}}{(1-\lambda\cos(\omega))^{1/2}}\right)_{,\tau}+\hat{t}_{01}\left(\frac{\mu_2'(0)}{(1-\lambda)^{1/2}}\right)\eta_{,\hat{t}\hat{t}}^{11}+T_0\left(\frac{\mu_2'(\tau)\eta_{,\hat{t}}^{11}}{(1-\lambda\cos(\omega))^{1/2}}\right)_{,\tau}$$
$$+\frac{\mu_2'(0)\eta_{,\hat{t}}^{11}}{(1-\lambda)^{1/2}}+\frac{\hat{t}_{21}\eta_{,\hat{t}\tau}^{10}}{(1-\lambda)^{1/2}}+\frac{\hat{t}_{20}\eta_{,\hat{t}\tau}^{11}}{1-\lambda)^{1/2}}+\frac{\eta_{,\tau}^{31}}{(1-\lambda)^{1/2}}=0 \qquad (3.32g)$$

On solving (3.32*a*). *that is*,

$\eta_{,\hat{t}}^{10}=0$ *and noting* $\eta^{10}(\hat{t},\tau)=a_{10}(\tau)cos\hat{t}+B(\tau)$

Hence,

$$\eta_{\hat{t}}^{10}(\hat{t}_{0,}0)=-a_{10}(0)sin\hat{t}_0=0; sin\hat{t}_0=0\ but\ a_{10}(0)\neq 0; \therefore \hat{t}_0=n\pi; n=0,1,2,\ldots$$

Since the aim is to look for a nontrivial solution, it is here necessary to take the *smallest non − trivial value of n*; *which is* $n=1$, *and obtain*

$$\hat{t}_0=\pi(3.32h)$$

The Solution of (3.32*b*) *is*

$$\hat{t}_{01}=\frac{1}{\eta_{,\hat{t}\hat{t}}^{10}}\left[T_0\eta_{,\hat{t}\tau}^{10}+\eta_{,\hat{t}}^{11}+\frac{\eta_{,\tau}^{10}}{(1-\lambda)^{1/2}}\right](\hat{t}_{0,}0)$$

Since $\eta_{,\hat{t}\tau}^{10}(\hat{t}_{0,}0)=b_{11}(0)=\eta_{,\tau}^{10}=0$

$\therefore \hat{t}_{01}=0$ (3.32i)

It is noted that,

$$\eta^{11}(\hat{t}_{0,}0)=b_{11}(0)cos\hat{t}_0=0\,; b_{11}(0)cos\hat{t}=0\,; b_{11}(0)=0$$

Solving (3.32*d*), *gives*

$\hat{t}_{10}\eta_{\hat{t}\hat{t}}^{10}(t_{0,}0)=0; \therefore \hat{t}_{10}=0\ but\ \eta_{,\hat{t}\hat{t}}^{10}\neq 0, also\ \hat{t}_{02}=\hat{t}_{10}=\hat{t}_{11}=\hat{t}_{12}=\hat{t}_{20}=\hat{t}_{21}=\hat{t}_{22}=0$

The summary of the displacement so far is as follows;

$$\eta(\hat{t},\tau) = \epsilon(\eta^{10} + \delta\eta^{11} + \delta^2\eta^{12} + \cdots) + \epsilon^3(\eta^{30} + \delta\eta^{31} + \delta^2\eta^{32} + \cdots) + \cdots \quad (3.33)$$

Let $\eta_C = \eta(\hat{t}_c, \tau_c)$ *be the maxmium of* $\eta(\hat{t},\tau)$ *and from* (3.33), *it is evident that,*

$$\eta_c = \eta(\hat{t}_{c,}\tau_c) = \epsilon(\eta_c^{10} + \delta\eta_c^{11} + \delta^2\eta_c^{12} + \cdots) + \epsilon^3(\eta_c^{30} + \delta\eta_c^{31} + \delta\eta_c^{32} + \cdots) + \cdots \quad (3.34)$$

where;

$$\eta_c{}^{ij} = \eta^{ij}(\hat{t}_{c,}\tau_c) \quad (3.35)$$

Expanding each term of (3.35) *asymptotically, gives the following*

$$\epsilon\eta_c^{10} = \epsilon\big[\eta^{10}(\hat{t}_0 0) + \eta_{,\hat{t}}^{10}\{(\delta\hat{t}_{01} + \delta^2\hat{t}_{02} + \cdots) + \epsilon(\hat{t}_{10} + \delta\hat{t}_{11} + \delta^2\hat{t}_{12} + \cdots)$$
$$+ \epsilon^2(\hat{t}_{20} + \delta\hat{t}_{21} + \delta^2\hat{t}_{22} + \cdots)\} + \eta_{\cdot\tau}^{10}\,\delta\{(T_0 + \delta T_{01} + \cdots) + \epsilon(T_{10} + \delta T_{11} + \cdots)$$
$$+\epsilon^2(T_{20} + \delta T_{21} + \cdots)\} + \frac{1}{2}\{\eta_{.\hat{t}\hat{t}}^{10}\{\delta\hat{t}_{01} + \delta^2\hat{t}_{02} + \cdots\} + \epsilon(\hat{t}_{01} + \delta\hat{t}_{11} + \delta^2\hat{t}_{12} + \cdots)$$
$$+\epsilon^2(\hat{t}_{20} + \delta\hat{t}_{21} + \delta^2\hat{t}_{22} + \cdots)\}^2 + \delta\eta_{\cdot\hat{t}\tau}^{10}\{(T_0 + \delta T_{11} + \cdots) + \epsilon(T_{10} + \delta T_{11} + \cdots)$$
$$+\epsilon^2(T_{20} + \delta T_{21} + \cdots)\}\{\delta\hat{t}_{01} + \cdots + \epsilon(\hat{t}_{01} + \delta\hat{t}_{01} + ..) + \epsilon^2(\hat{t}_{20} + \delta\hat{t}_{21} + \cdots)\}$$
$$+\delta^2\eta_{,\tau\tau}^{10}\{(T_0 + \cdots) + \epsilon(T_{10} + \cdots) + \epsilon^2(T_{20} + \cdots)\}\big] \quad (3.36)$$

$$\epsilon\delta\eta_c^{11} = \epsilon\delta\big[\eta^{11}(\hat{t}_{0,}0) + \eta_{,\hat{t}}^{11}\{(\delta\hat{t}_{01} + \cdots) + \epsilon(\hat{t}_{10} + \delta\hat{t}_{11} + \cdots) + \epsilon^2(\hat{t}_{20} + \delta\hat{t}_{21} + \cdots)\}$$
$$+\eta_{,\tau}^{11}\delta\{(T_0 + \cdots) + \epsilon(T_{10} + \cdots) + \epsilon^2(T_{20} + \cdots)\} + \frac{1}{2}\{\eta_{,\hat{t}\hat{t}}^{11}\{(\delta\hat{t}_{01} + \cdots)\} + \epsilon(\hat{t}_{10} + \delta\hat{t}_{11} + \cdots)$$
$$+\epsilon^2(\hat{t}_{20} + \delta\hat{t}_{21} + \cdots)\}^2 + 2\delta\eta_{,\hat{t}\tau}^{10}\{(\hat{t}_0 + \cdots) + \epsilon(\hat{t}_{10} + \cdots) + \epsilon^2(\hat{t}_{20} + \cdots)\}$$
$$+\delta^2\eta_{,\tau\tau}^{12}\{(T_0 + \ldots) + \epsilon(T_{10} + \cdots) + \epsilon(T_{20} + \cdots)\}\big] \quad (3.37)$$

$$\epsilon\delta^2\eta_c^{12} = \epsilon\delta^2\big[\eta^{12}(\hat{t}_0, 0) + \eta_{,\hat{t}}^{12}\{\ldots + \epsilon(\hat{t}_{10} + \cdots) + \epsilon^2(\hat{t}_{20} + \cdots)\}\big] \quad (3.38)$$

$$\epsilon^3\eta_c^{30} = \epsilon^3\big[\eta^{30}(\hat{t}_0, 0) + \eta_{,\hat{t}}^{30}\{(\delta\hat{t}_{01} + \delta^2\hat{t}_{02} + \cdots)\} + \delta\eta_{,\tau}^{30}\{(T_0 + \delta T_{01} + \cdots) + \cdots\}\big] \quad (3.39)$$

$$\epsilon^3\delta\eta_c^{31} = \epsilon^3\delta\big[\eta^{31}(\hat{t}_{0,}0) + \eta_{,\tau}^{31}(\delta\hat{t}_{01} + \cdots) + \delta\eta_{,\tau}^{31}\{(T_0 + \cdots) + \cdots\}\ \big] \quad (3.40)$$

$$\epsilon^3\delta^2\eta_c^{32} = \epsilon^3\delta^2[\eta^{32}(\hat{t}_0, 0) + \cdots] \quad (3.41)$$

Grouping (3.36) *to* (3.41) *in orders* $(\epsilon^i\delta^j)$, *results to*

$$\eta_c = \epsilon\Bigg[\eta^{10}(\hat{t}_{0,}0) + \delta\{\hat{t}_{0,}\eta_{,\hat{t}}^{10} + \eta^{11}(\hat{t}_{0,}0) + T_0\eta_{,\tau}^{10}\} + \delta^2\Bigg\{\hat{t}_{02}\eta_{,\hat{t}}^{10} + T_{01}\eta_{,\tau}^{10} + \frac{\hat{t}_{01}}{2}\eta_{,\hat{t}\hat{t}}^{10}$$
$$+T_0\hat{t}_{01}\eta_{\hat{t}\tau}^{10} + \frac{T_0^2}{2}\eta_{,\tau\tau}^{10} + \hat{t}_{01}\eta_{,\hat{t}}^{11} + T_0\eta_{,\tau}^{11} + \eta^{12}(\hat{t}_{0,}0) + \Bigg\}\Bigg] + \cdots + \epsilon^3\big[\hat{t}_{20}\eta_{,\hat{t}}^{10}$$
$$+\frac{\hat{t}_{01}}{2}\eta_{,\hat{t}\hat{t}}^{10} + \eta^{32}(\hat{t}_{0,}0) + \delta\{\hat{t}_{21}\eta_{,\hat{t}}^{10} + T_{20}\eta_{,\tau}^{10} + \hat{t}_{11}\eta_{,\hat{t}}^{10} + T_0\hat{t}_{20}\eta_{,\hat{t}\tau}^{10} + \hat{t}_{20}\eta_{,\hat{t}}^{11}$$
$$+\frac{\hat{t}_{10}}{2}\eta_{,\hat{t}\hat{t}}^{11} + \hat{t}_{01}\eta_{,\hat{t}}^{30} + T_0\eta_{,\tau}^{30} + \eta^{31}(\hat{t}_{0,}0) + \delta^2\{\hat{t}_{22}\eta_{,\hat{t}}^{10} + T_{21}\eta_{,\tau}^{10} + \frac{1}{2}\{\{2\hat{t}_{01}\hat{t}_{21}$$
$$+2\hat{t}_{02}\hat{t}_{20} + \eta^{31}(\hat{t}_0, 0) + \hat{t}_{11}\}\}\eta_{,\hat{t}\hat{t}}^{10} + \{2(\hat{t}_{01}T_{20} + T_0\hat{t}_{21} + \hat{t}_{10}T_{11} + T_0\hat{t}_{11})\}\eta_{,\hat{t}\tau}^{30}$$
$$+T_{10}^2\eta_{,\tau\tau}^{2} + \hat{t}_{21}\eta_{,\hat{t}}^{11} + T_{20}\eta_{,\tau}^{11} + (\hat{t}_{01}\hat{t}_{20} + \hat{t}_{10}\hat{t}_{11})\eta_{,\hat{t}\hat{t}}^{11} + (\hat{t}_0\hat{t}_{20} + \hat{t}_{10}^2)\eta_{,\hat{t}\tau}^{11} + \hat{t}_{20}\eta_{,\hat{t}}^{12}$$
$$+\hat{t}_{02}\eta_{,\hat{t}}^{30} + T_{01}\eta_{,\tau}^{30} + \hat{t}_{01}\eta_{,\hat{t}}^{31} + T_0\ \eta_{,\tau}^{30} + \eta^{32}(\hat{t}_{0,}0)\}\big] \quad (3.42)$$

Further simiplifcation of (3.42) *gives*

$$\eta_c = \epsilon\left[\eta^{10} + \delta^2\left(\frac{T_0^2}{2}\eta_{,\tau\tau}^{10} + T_0\eta_{,\tau}^{11} + \eta^{12}\right)\right](\hat{t}_{0,}0)$$
$$+\epsilon^3\big[\eta^{30} + \delta(T_0\eta_{,\tau}^{30} + \eta^{31}) + \delta^2(T_{20}\eta_{,\tau}^{11} + T_{01}\eta_{,\tau}^{30} + T_0\eta_{,\tau}^{31} + \eta^{32})\big](\hat{t}_{0,}0) + \cdots \quad (3.43a)$$

Substituting the values of $\eta^{ij}(\hat{t}_{0,}0)$ *in* (3.43*a*), *gives*

$$\eta_c = \epsilon\left[2B(0) + \delta^2\left\{\frac{T_0^2}{2}\left(\omega'^2(0)B(0) - \omega'^2(0)\right) + \frac{2\omega'^2(0)}{(1-\lambda)}\right\}\right]$$
$$+\epsilon^3\left[\frac{67B^3(0)b}{32(1-\lambda)} - \frac{6\delta^2 b}{(1-\lambda)^3}\left(\omega'^2(0)B(0) - 2\omega'^2(0)B(0)\right)\right] + \cdots \quad (3.43b)$$

Equation(3.43*b*) *contains the term* T_0^2 *which is yet to be determined. This term is now determined with*recourse to (2.20*c*)

$$Thus\ at\ maxmium\ displacement, the\ expansion\ of\ \ (2.20c)\ gives$$

$$\hat{t}_c = \tilde{t}_c + \frac{1}{\delta}\{\mu_2(\tau)\epsilon^2 + \mu_3(\tau)\epsilon^3 + \cdots\} \qquad (3.44a)$$

Let,

$$\tilde{t}_c = \tilde{t}_0 + \delta\,\tilde{t}_{01} + \delta^2\tilde{t}_{02} + \cdots + \epsilon(\tilde{t}_{10} + \delta\tilde{t}_{11} + \delta^2\tilde{t}_{12} + \cdots) + \epsilon^2(\tilde{t}_{20} + \delta\tilde{t}_{21} + \cdots) + \cdots \quad (3.44b)$$

Expansion $(4.44a)$, *gives*

$$\hat{t}_c = \tilde{t}_c + \frac{\epsilon^2}{\delta}\left[\mu_2(0) + \mu_2'(0)(\delta t_c) + \frac{1}{2}\mu_2''(0)(\delta t_c)^2 + \cdots\right]; \quad \mu_2(0) = 0 \qquad (3.44c)$$

$$\therefore \hat{t}_c = \tilde{t}_c + \epsilon^2\left(\mu_2'(0)t_c + \frac{1}{2}\mu_2''(0)\delta t_c^2 + \cdots\right) \qquad (3.44d)$$

Expansion of both sides of $(3.44d)$, *gives*

$$\hat{t}_0 + \delta\tilde{t}_{01} + \delta^2\hat{t}_{02} + \cdots + \epsilon(\hat{t}_{10} + \delta\hat{t}_{11} + \delta^2\hat{t}_{12} + \cdots) + \epsilon^2(\hat{t}_{20} + \delta\hat{t}_{21} + \delta^2\hat{t}_{22} + \cdots)$$
$$= \tilde{t}_0 + \delta\tilde{t}_{01} + \delta^2\tilde{t}_{02} + \cdots + \epsilon(\tilde{t}_{10} + \delta\tilde{t}_{11} + \delta^2\tilde{t}_{12} + \cdots) + \epsilon^2(\tilde{t}_{20} + \delta\tilde{t}_{21} + \delta^2\tilde{t}_{22} + \cdots)$$
$$+ \epsilon^2\left[\left(\mu_2'(0)\right)\{\tilde{t}_0 + \delta\tilde{t}_{01} + \cdots + \epsilon^2(\tilde{t}_{20} + \delta\tilde{t}_{21} + \delta\tilde{t}_{22} + \cdots) + \cdots\}\right]$$
$$+ \frac{\mu_2''(0)\delta}{2}[\tilde{t}_0 + \delta\tilde{t}_{01} + \delta^2\tilde{t}_{02} + \cdots + \epsilon(\tilde{t}_{10} + \delta\tilde{t}_{11} + \cdots) + \epsilon^2(\tilde{t}_{20} + \delta\tilde{t}_{21} + \cdots)]^2 \qquad (3.45)$$

On equating the orders of both sides, the result gives

$$O(1): \hat{t}_0 = \tilde{t}_0; O(\delta): \hat{t}_{01} = \tilde{t}_{01}; O(\delta^2): \hat{t}_{02} = \tilde{t}_{02}; O(\epsilon): \hat{t}_{10} = \tilde{t}_{10} \qquad (3.46a,b,c,d)$$

$$O(\epsilon^2): \hat{t}_{20} = \tilde{t}_{20} + \mu_2'(0)\tilde{t}_0 \qquad (3.46e)$$

It is to be recalled from $(2.20b)$ *that,*

$$\frac{d\tilde{t}}{dt} = [1 - \lambda\cos(\omega(\delta t)]^{1/2} \qquad (3.47)$$

Expansion of the right hand side of (3.47), *gives*

$$\frac{d\tilde{t}}{dt} = \left[1 - \lambda\left\{1 - \frac{\omega^2(\tau)}{2!} + \frac{\omega^4(\tau)}{4!} + \cdots\right\}\right]^{1/2} = \left[(1-\lambda) + \lambda\left\{\frac{\omega^2(\tau)}{2} - \frac{\omega^4(\tau)}{4!} + \cdots\right\}\right]^{1/2} \quad (3.48a)$$

Further simplification of $(4.48a)$ *using binomial expansion gives*

$$\frac{d\tilde{t}}{dt} = (1-\lambda)^{1/2}\left[1 + \frac{1}{2}\left(\frac{\lambda}{1-\lambda}\right)\left\{\frac{\omega^2(\tau)}{2!} - \frac{\omega^4(\tau)}{4!} + \cdots\right\} - \frac{1}{8}\left(\frac{\lambda}{1-\lambda}\right)^2\left\{\frac{\omega^2(\tau)}{2!} - \frac{\omega^4(\tau)}{4!} + \cdots\right\}^2\right] (3.48b)$$

More expansion of $(4.48b)$, *gives*

$$\frac{d\tilde{t}}{dt} = (1-\lambda)^{1/2}\left[1 + \frac{1}{2}\left(\frac{\lambda}{1-\lambda}\right)\left\{\left\{\frac{\omega^2(0)}{2} + \frac{(\omega^2)'(0)}{1!}\delta t + \frac{(\omega^2)''(0)}{2!}(\delta t)^2 + \cdots\right\}\right\}\right.$$
$$\left. - \frac{1}{8}\left(\frac{\lambda}{1-\lambda}\right)^2\frac{1}{2}\left\{\omega^2(0) + \frac{(\omega^2)'(0)}{1!}\delta t + \frac{(\omega^2)''\ \ (0)}{2!}(\delta t)^2 + \cdots\right\}\right]$$
$$- \frac{1}{4!}\left[\omega^4(0) + \frac{(\omega^4)'(0)\delta t}{1!} + \frac{(\omega^4)''^2(0)(\delta t)^2}{2!} + \cdots\right] - \frac{1}{4!}\left\{\omega^4(0) + \frac{\omega'^4(0)\delta t}{1!} + \frac{\omega''^4(0)(\delta t)^2}{2!} + ..\right\}^2 \quad (3.48c)$$

Further simplification $(3.48c)$ *gives*

$$\frac{d\tilde{t}}{dt} = (1-\lambda)^{1/2}\left[\left\{1 + \frac{\lambda}{2(1-\lambda)}\left\{\frac{1}{2}\omega^2(0) - \frac{1}{4!}\omega^4(0)\right\}\right\} - \frac{1}{8}\left(\frac{\lambda}{1-\lambda}\right)^2\left\{\frac{1}{4}\omega^4(0) - \frac{\omega^8(0)}{(4!)^2}0\right\}\right.$$
$$\left. + \left\{\frac{-\lambda}{2(1-\lambda)}\left(\frac{(\omega^2)'(0)}{2} - \frac{(\omega^4)'(0)}{4!}\right) - \frac{1}{8}\left(\frac{\lambda}{1-\lambda}\right)^2\right\}\right] + \left\{\frac{\omega^2(0)\omega'^2(0)}{2} - \frac{\omega^4(0)\omega'^4(0)}{288}\right.$$
$$\left. - \frac{\omega^2(0)\omega'^4(0)}{1152} - \frac{\omega'^2(0)\omega^4(0)}{1152}\right\}\delta t + \left\{\left\{-\frac{1}{8}\left(\frac{\lambda}{1-\lambda}\right)^2\left\{\frac{(\omega^2\ \ )'(0)}{3} + \frac{\omega^2(0)(\omega^2)''(0)}{2}\right\} - \left(\frac{(\omega^4)'(0)}{576}\right)^2\right.\right.$$

$$-\frac{\omega^3(0)\omega''^4(0)}{576}-\frac{\omega^2(0)\omega''^4(0)}{24}-\frac{\omega'^2(0)\omega^2(0)}{24}-\frac{\omega''^2(0)\omega^3(0)}{48}\}\}(\delta t)^2+\cdots \qquad (3.48d)$$

Now (3.48*d*) *can be written simply as*

$$\frac{d\tilde{t}}{dt}=(1-\lambda)^{1/2}[\beta_1+\beta_2\delta t+\beta_3(\delta t)^2+\cdots] \qquad (3.49)$$

where,

$$\beta_1=\left[1+\frac{\lambda}{2(1-\lambda)}\left\{\frac{\omega^2(0)}{2}-\frac{\omega^4(0)}{24}\right\}-\frac{1}{8}\left(\frac{\lambda}{1-\lambda}\right)^2\left\{\frac{\omega^4(0)}{4}-\frac{\omega^8(0)}{576}\right\}\right] \qquad (3.50a)$$

$$\beta_2=\left[\frac{-\lambda}{2(1-\lambda)}\left\{\frac{(\omega^2)'(0)}{2}-\frac{(\omega^4)'(0)}{576}\right\}-\frac{1}{8}\left(\frac{\lambda}{1-\lambda}\right)^2\left\{\frac{\omega^2(0)(\omega^2)'(0)}{2}-\frac{\omega^4(0)(\omega^4)'(0)}{288}\right\}-\frac{\omega^2(0)\omega'(0)}{1152}\right] \qquad (3.50b)$$

$$\beta_3=-\frac{1}{8}\left(\frac{\lambda}{1-\lambda}\right)^2\left[\frac{(\omega^2)'(0)}{3}+\frac{\omega^2(0)(\omega^2)''(0)}{2}-\frac{(\omega^4)'(0)}{576}-\frac{\omega^4(0)(\omega^4)''(0)}{576}-\frac{(\omega^2)'(0)(\omega^4)''(0)}{24}-\frac{(\omega^2)''(0)\omega^4(0)}{488}\right] \qquad (3.50c)$$

From (3.49), it follows that,

$$d\tilde{t}=(1-\lambda)^{1/2}[\beta_1+\beta_2\delta t+\beta_3(\delta t)^2+\cdots]dt \qquad (3.51a)$$

integrating both sides of (3.51*a*), *gives*

$$\tilde{t}=(1-\lambda)^{1/2}\left[\beta_1 t+\frac{1}{2}\beta_2\delta t^2+\frac{\beta_3}{3}\delta^2 t^3+\cdots\right] \qquad (3.51b)$$

Thus, using (3.51*b*), *it follows that,*

$$\tilde{t}_c=(1-\lambda)^{1/2}\left[\beta_1 t_c+\frac{\beta_2\delta t_c^2}{2}+\frac{\beta_3\delta^2 t_c^3}{3}+\cdots\right] \qquad (3.51c)$$

The term $\tilde{t}_c$ *can be expanded as*

$$\tilde{t}_c=(\tilde{t}_0+\delta\tilde{t}_{01}+\delta^2\tilde{t}_{02}+\cdots+\epsilon(\tilde{t}_{10}+\delta\tilde{t}_{11}+\delta^2\tilde{t}_{12}+\cdots)+\epsilon^2(\tilde{t}_{20}+\delta\tilde{t}_{21}+\cdots)+\cdots$$

While it is to be recalled from(3.29b) that

$$t_c=T_0+\delta T_{01}+\delta^2 T_{02}+\cdots+\epsilon(T_{10}+\delta^2 T_{12}+\cdots)+\epsilon^2(T_{20}+\delta T_{21}+\delta^2 T_{22}+\cdots)+\cdots$$

By asymptotic expansion of both sides of (3.51*c*), *the result gives*

$$\tilde{t}_0+\delta\tilde{t}_{01}+\delta^2\tilde{t}_{02}+\cdots+\epsilon(\tilde{t}_{10}+\delta\tilde{t}_{11}+\delta^2\tilde{t}_{12}+\cdots)+\epsilon^2(\tilde{t}_{20}+\delta\tilde{t}_{21}+\delta^2\tilde{t}_{22}+\cdots)+$$
$$=(1-\lambda)^{1/2}[\beta_1\{T_0+\delta T_{01}+\delta^2 T_{02}+\cdots+\epsilon(T_{10}+\delta T_{11}+\delta^2 T_{12}+\cdots)+\epsilon^2(T_{20}+\delta T_{21}$$
$$+\delta^2 T_{22}+\cdots)\}]+\frac{\beta_{2\delta}}{2}[(T_0+\delta T_{01}+\delta^2 T_{02}+\cdots)+\cdots+\epsilon(T_{10}+\delta T_{11}+\delta^2 T_{12}+\cdots)$$
$$+\epsilon^2(T_{20}+\delta T_{21}+\delta^2 T_{22}+\cdots)]^2+\frac{\beta_3\delta^2}{3}[(T_0+\delta T_{01}+\cdots)$$
$$+\epsilon(T_{10}+\delta T_{11}+\cdots)+\epsilon^2(T_{20}+\delta T_{22}+\cdots)]^3 \qquad (3.51d)$$

On equating the orders of both sides of (4.51*d*), *the following are obtained*

$$O(1):\tilde{t}_0=(1-\lambda)^{1/2}\beta_1 T_0;\ O(\delta):\tilde{t}_{01}=(1-\lambda)^{1/2}\left[\beta_1 T_{01}+\frac{\beta_2 T_0}{2}\right] \qquad (3.52a,b)$$

$$O(\delta^2):\tilde{t}_{02}=(1-\lambda)^{1/2}[\beta_1 T_{02}+\beta_2 T_0 T_{01}];\ O(\epsilon):\tilde{t}_{10}=(1-\lambda)^{1/2}\beta_1 T_{10} \qquad (3.52c,d)$$

$$O(\epsilon\delta):\tilde{t}_{11}=(1-\lambda)^{1/2}[\beta_1 T_{11}+\beta_2 T_0 T_{01}] \qquad (3.52e)$$

$$O(\epsilon\delta^2):\tilde{t}_{12}=(1-\lambda)^{1/2}[\beta_1 T_{12}+\beta_2(T_0 T_{11}+T_{01}T_{10})];\ O(\epsilon^2):\tilde{t}_{20}=(1-\lambda)^{\frac{1}{2}}\beta_1 T_{20} \qquad (3.52f,g)$$

$$O(\epsilon^2\delta):\tilde{t}_{21}=(1-\lambda)^{1/2}\left[\beta_1 T_{21}+\frac{\beta_2}{2}\left(T_{10}^{\ 2}+2T_{10}T_{20}\right)\right] \qquad (3.52h)$$

$$O(\epsilon^2\delta^2):\tilde{t}_{22}=(1-\lambda)^{1/2}[\beta_1 T_{22}+\beta_2(T_0 T_{21}+T_{10}+T_{11}+T_0 T_{21}+T_{01}T_{20})] \qquad (3.52I)$$

Substituting from (3.46*a*), (3.32*b*), $\hat{t}_0=\pi$ *in* (3.52*a*), *gives,*

$$T_0 = \frac{\pi}{\beta_1(1-\lambda)^{1/2}}, also, T_{01} = \frac{\beta_2\pi^2}{2\beta_1^3(1-\lambda)}; T_{02} = -\frac{\beta_2\pi^3}{2\beta_1^5(1-\lambda)^{5/2}} \quad (3.53a,b,c)$$

$From\ (3.52d), it\ follows\ that,$

$$T_{10} = 0; since\ \tilde{t}_{10} = 0, also, T_{11} = T_{12} = T_{20} = T_{21} = T_{22} = 0$$

$Hence, we\ have\ determined\ T_0, T_{01}\ and\ T_{20}\ which\ are\ to\ be\ substituted\ in\ (3.43a)$

$On\ expansion\ of\ (3.43b), the\ results\ gives$

$$\eta_c = \left[2B(0) + \delta^2\left\{\frac{T_0^2}{2}\left(\frac{\omega'^2(0)B(0) - \omega'^2(0)}{(1-\lambda)}\right) + \frac{2\omega'^2(0)}{(1-\lambda)}\right\}\right] + \epsilon^2\left[\frac{67B^2(0)b}{32(1-\lambda)} - \frac{6\delta^2 b}{(1-\lambda)^3}\left\{\omega'^2(0)B(0) - 2B^2(0)\omega'^2(0)\right\}\right]$$

$This\ can\ be\ recasted\ as$

$$\eta_c = \epsilon G_1 + \epsilon^2 G_2 + \cdots \quad (3.54)$$

$where,$

$$G_1 = 2B(0)\left[1 + \frac{\delta^2\omega'^2(0)}{4(1-\lambda)B(0)}\{B(0) - 1)T_0^2 + 4\}\right] \quad (3.55a)$$

$$G_2 = \frac{67B^3(0)b}{32(1-\lambda)}\left[1 + \frac{192\delta^2\omega'^2(0)(2B(0) - 1)}{67B^2(0)(1-\lambda)^2}\right] \quad (3.55b)$$

$Further\ simplification\ (3.55a, b)\ gives;$

$$G_1 = 2B(0)(1 + d_1\delta^2); G_2 = \frac{67B^3(0)b}{32(1-\lambda)}(1 + d_2\delta^2) \quad (3.56a,b)$$

$where,$

$$d_1 = \frac{\omega'^2(0)}{4(1-\lambda)B(0)}\{(B(0) - 1)T_0^2 + 4\};\ \ d_2 = \frac{192\omega'^2(0)}{67B^2(0)(1-\lambda)^2}(2B(0) - 1) \quad (3.57a,b)$$

3.4 Dynamic buckling load of the model structure

$According\ to\ Amazigo\ and\ Ette\ (1987), the\ procedure\ for\ finding\ the\ dynamic\ buckling\ load\ \lambda_D,\ involves\ first\ reversing\ the\ series\ in\ (3.54) as\ follows$:

$$\epsilon = k_1\eta_c + k_3\eta_c^3 + \cdots \quad (3.58)$$

Such a reversal of series is required in order to ensure the boundedness of solution after buckling

$Substituting\ for\ \eta_c in\ (3.58)\ using\ (3.54)\ results\ to$

$$\epsilon = k_1(\epsilon G_1 + \epsilon^3 G_3 + \cdots) + k_3(\epsilon G_1 + \epsilon^3 G_3 + \cdots)^3 + \cdots \quad (3.59)$$

$Equating\ the\ coefficients\ of\ \epsilon\ in\ (3.59), leads\ to$

$$k_1 G_1 = 1. \therefore k_1 = \frac{1}{G_1} \quad (3.60a)$$

$Equating\ the\ coefficient\ of\ \epsilon^3 in\ (3.59), results\ to$

$$k_1 G_3 + k_3 G_1^3 = 0. \therefore k_3 = -\frac{k_1 G_3}{G_1^3} = -\frac{G_3}{G_1^4} \quad (3.60b)$$

$To\ obtain\ the\ dynamic\ buckling\ load\ \lambda_D, it\ is\ necessary\ to\ invoke (3.26)\ and\ obtain$

$$\frac{d\epsilon}{d\eta_c} = \frac{d}{d\eta_c}[k_1\eta_c + k_3\eta_c^3 + \cdots]; \frac{d\epsilon}{d\eta_c} = 0; \epsilon\ (constant) \quad (3.61a)$$

$$\therefore \frac{d(k_1)}{d\lambda}\frac{d\lambda}{d\eta_c}\eta_c + k_1\frac{d\eta_c}{d\eta_c} + \frac{d(k_3)}{d\lambda}\frac{d\lambda}{d\eta_c}\eta_c^3 + \frac{k_3 d(\eta_c^3)}{d\eta_c} = 0 \quad (3.61b)$$

$From\ (3.61b), it\ follows\ that$

$$k_1 + 3k_3\eta_c^2 + \cdots = 0; \left(since\ \frac{d\lambda}{d\eta_c} = 0\right); \lambda = \lambda_D; and\ k_1(\lambda_D) + 3k_3(\lambda_D)\eta_{CD}^2 = 0 \quad (3.61c,d)$$

$where,$

$$\eta_{CD} = \eta_c(\lambda_D).\ \therefore \eta_{CD} = \left(\frac{-k_1(\lambda_D)}{3k_3(\lambda_D)}\right)^{1/2} = \left(\frac{-k_1}{3k_3}\right)^{1/2} \quad (3.61e)$$

$Substituting\ the\ values\ of\ k_1 and\ k_3 respectively\ in\ (3.61e), leads\ to$

$$\eta_{CD} = \left(\frac{G_1^3}{3G_3}\right)^{1/2} \quad (3.61f)$$

where, η_{CD} is the displacement at buckling. To determine the dynamic buckling load λ_D, there is need to evaluate (3.58) at buckling and get,

$$\epsilon = k_1(\lambda_D)\eta_{CD} + k_3(\lambda_D)\eta_{CD}^3 + \cdots \quad (3.62)$$

$Substituting\ the\ values\ of\ k_1 and\ k_3 and\ \eta_{CD} in\ (3.62), results\ to$

$$\epsilon = \frac{2}{3\sqrt{3}}\sqrt{\frac{G_1}{G_3}};\ \epsilon = \frac{2}{3\sqrt{3}}\left[\frac{2B(0)}{\frac{67B^2(0)b}{32(1-\lambda)}}\left(\frac{1+d_1\delta^2}{1+d_2\delta^2}\right)\right]^{1/2} \quad (3.63a,b)$$

$Further\ simplifcation\ of\ (3.63b) gives$

$$(1-\lambda_D)^{3/2} = \frac{3\sqrt{201}}{16}\lambda_D b^{1/2}\epsilon\ Z^{1/2}\ ;\text{and } Z = \left(\frac{1+d_2\delta^2}{1+d_1\delta^2}\right) \quad (4.64)$$

CONCLUSION

Hence, we see that the dynamic buckling load, λ_D depends on the first derivative of the circular frequency evaluated at the initial time. We found that the least order of dependence of λ_D on δ is of the form δ^2 (4.64). Hence, if is very small compared to unity $i.e. 0 < \delta << 1,$ then δ^2 is even smaller and generally tends to zero. Thus, periodic loading with slowly varying circular frequency tends to a step load. The equation (4.46) is the expected load equation and λ_D is dynamic buckling load.

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