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ANALYSIS OF AN ISOTROPIC HOMOGENEOUS ELASTIC MATERIAL CONTAINING A FINITE INHOMOGENEITY AT THE END OF A CRACK UNDER ANTIPLANE SHEAR.

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ABSTRACT

In this paper, we consider the determination of the mode III stress intensity factor (SIF) at the tips of a finite line inhomogeneity (anti-crack) of length b units lying on the right-hand side of the x-axis in an infinite elastic material with loads T and Q applied on the surface of the crack at lengths l and h respectively from the origin. The inhomogeneity is rigidly bonded and so is displacement free. The loading gives rise to two-dimensional boundary value problem for a Laplace equation that models the antiplane strain displacement $w(r,\theta)$. The problem is solved by Mellin integral transform and Wiener-Hopf technique. The displacement and stress fields were obtained leading to the stress intensity factors k_{III}^{outer} and k_{III}^{inner} for the deformation at the outer inhomogeneity tip and at its inner tip respectively. The existence of the stress intensity factors implies that crack initiation can start either at the outer or inner tip depending on the loading. A linear relationship is found between k_{III}^{nor} and $\frac{l}{b}$ where k_{III}^{nor} is the normal stress intensity factor formed by the ratio of k_{III}^{inner} to the known mode III stress intensity factor k_{III}^{0} at the tip of a crack in a material of the same geometry as the one being investigated. A similar relationship is found also for k_{III}^{outer} as shown in the graph.

1. Introduction

Analytical solutions of elastic problems for structures comprising of a crack and a perfectly bonded rigid line inhomogeneity is of special interest in corresponding failure analysis. In recent years, the failure analysis of such configuration has reinvigorated the interest of researchers. This may be attributed to mainly to the increasing applications of such structures by wielders, motor mechanics and panel beaters. inhomogeneities of any form usually alter the elastic response of a material as well as its fracture properties

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A rigid line inhomogeneity embedded in an elastic material is of theoretical interest because it is the counterpart of conventional crack in solids, hence the understanding of the characteristics of the elastic fields at a rigid line inhomogeneity becomes very imperative. The characteristics of the stress field near the tips of a rigid line inhomogeneity have been carried out by Chou [1] and Wang *et al* [2] . In their works, they obtained analytical solution to the problem of a rigid line inhomogeneity under the action of inclined loading. Employing Eshelby's equivalent inclusion method and conformal mapping of Mushelishvili's complex potentials, they obtained the stresses at the tips of the inhomogeneity which has square root singularity. The stress intensity factors derived is seen to depend on the Poisson's ratio.

Xian *et al* [3] solved the problem of interaction of tip fields between periodic cracks and periodic rigid line inclusions. Employing exact solution methods, they obtained a closed form expression for the stress intensity factor (SIF) at the tips of crack and rigid line inclusions. Their result depicts that (i) the tip fields of cracks and rigid line inclusions show different laws when their horizontal and vertical distribution periods changes. (ii) With the increase of the length of crack, the SIF of cracks increases monotonously whereas the SIF of rigid line inclusions gradually decreases from 1 to 0, whereas the SIF of cracks is only slightly increased. A dislocation pileup model for micro-crack initiation at the inhomogeneity tip was proposed Xiao and Chen [4] based on Zener-Stroh crack initiation mechanism. The result of their analysis shows that the critical stress intensity factor for the anti-crack (line inhomogeneity) can be related to fracture toughness of a conventional Griffith crack in the same material. The analytical results further show that under mechanical loading, the stress and electric displacement intensity factors of an anti-crack are only related to the corresponding intensity factors of stress and electric displacement of the crack, respectively. Ballarini [5] presented an integral equation approach in solving rigid line inhomogeneity problem. This approach clearly illustrates the similarities between crack problems and rigid line inhomogeneity problems. The linear elastic plane deformation of a soft material containing a rigid line inhomogeneity subjected to a concentrated force, a concentrated moment and a point heat source was studied by Pengyu *et al* [6]. Using the Green function technique and solving the corresponding Riemann Hilbert problem, they obtained a closed form solution for the full stress field in the soft material. The problem studied here is an antiplane strain problem of an elastic infinite material having a semi-infinite crack and a rigid line inhomogeneity. Although presently related works has been carried out by many researchers, this work is distinguished from the previous as it went further to investigate the behavior of the SIF at the outer and inner tips of the inhomogeneity. The same technique was used by the author [7] to analyze an elastic homogeneous isotropic material with a right inhomogeneity embedded in the material under anti-shear The mathematical model of the problem was a boundary value problem formulated using the Mellin transform and solved by the Wiener-Hopf techniques. From his investigation, a closed formed solution for displacement was obtained from which the SIF was calculated. The stress field were found to have square root singularity at the inner tip inferring that a micro -cracking can initiate at the inner tip of the line inhomogeneity depending on the applied loads. The outer tip showed no singularity.

2.0 THREORETICAL ALALYSIS

[Mathematical Formulation]

Consider a homogenous elastic isotropic material occupying the region $-\infty < z < \infty$, $x = r\cos\theta, y = r\sin\theta, 0 \le r \le \infty, -\pi \le \theta \le \pi$ with a crack running from the left hand side of the x-axis and terminating at the origin, while on the positive x-axis there is a finite rigid inhomogeneity (anti-crack) of length b units, imbedded in the elastic material along $\theta = 0$ and $0 \le r \le b$. Anti-plane shear deformation state is achieved by the application of a pair of concentrated loads T and Q at the points of distance l and h from the origin. A ray $0 \le r \le b\ ,\theta = 0$ is shown in Fig. 1.

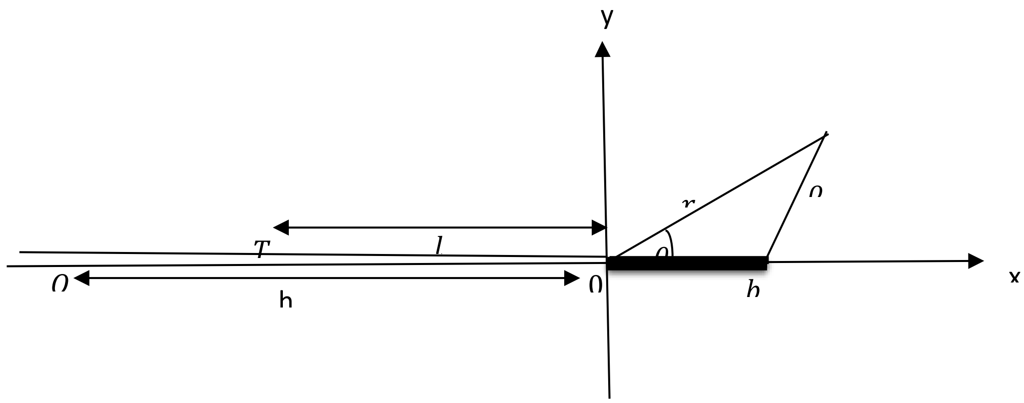

Fig.1. Geometry of the problem

By introducing the polar coordinate system (r,θ), we have the displacement $w(r,\theta)$ in z direction satisfying

$$\nabla^2 w(r,\theta) \equiv \left(\frac{\partial^2}{\partial r^2} z + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right) w(r,\theta) = 0\ , r \ge 0, -\pi \le \theta \le \pi \qquad (1)$$

The non-zero stresses are given by

$$\sigma_{\theta z} = \frac{\mu}{r}\frac{\partial w(r,\theta)}{\partial \theta}\ ,\ \sigma_{rz} = \mu \frac{\partial w(r,\theta)}{\partial r} \qquad (2a,b)$$

Where μ is the shear modulus of the material.

The boundary conditions are

$$w(r,0) = 0 \quad ,\ 0 \le r \le b \qquad (3a)$$

$$\sigma_{\theta z}(r,\pi) = T\delta(r-l) \quad , 0 \le \theta \le \pi \qquad (3b)$$

$$\sigma_{\theta z}(r,-\pi) = Q\delta(r-h) \quad , -\pi \le \theta \le 0 \qquad (3c)$$

Where δ is the Dirac's delta function

The continuity of the traction and displacement requires

$$w(r,0^+) = w(r,0^-) = 0 \qquad 0 \le r \le b \qquad (3d)$$

$$w(r,0^+) = w(r,0^-) \ne 0 \quad ,\ r > b \qquad (3e)$$

$$\sigma_{\theta z}(r,0^+) \ne \sigma_{\theta z}(r,0^-) \qquad 0 \le r \le b \qquad (3f)$$

$$\sigma_{\theta z}(r,0^+) = \sigma_{\theta z}(r,0^-) \qquad , r > b \qquad (3g)$$

The asymptotic behavior of the stresses is

$$\sigma_{\theta z,}\sigma_{rz} = \begin{cases} o\left(r^{-\lambda}\right) & as\ \ r \to 0 \\ 0\left(r^{-k}\right) & as\ \ r \to \infty \\ 0(r-b)^{-\frac{3}{2}} & as\ \ r \to b^+, \theta \to 0 \end{cases} \qquad (4a\text{-}c)$$

Defining the Mellin transform $\tilde{w}(s,\theta)$ of $w(r,\theta)$ as

$$\tilde{w}(s,\theta)=\int_0^\infty w(r,\theta)r^{s-1}dr\ ,\quad \lambda-1<\operatorname{Re}s<0 \tag{5}$$

Applying eqn.(5) to the governing equation eqn(1) and the boundary conditions (3) yields

$$\frac{d^2\tilde{w}(s,\theta)}{d\theta^2}+s^2\tilde{w}(s,\theta)=0 \qquad \lambda-1<\operatorname{Re}s<0,-\pi\le\theta\le\pi \tag{6a}$$

$$\frac{d\tilde{w}(s,0^+)}{d\theta}-\frac{d\tilde{w}(s,0^-)}{d\theta}=\frac{b^s}{\mu}G(s) \tag{6b}$$

$$\tilde{w}(s,0)=b^sF(s) \tag{6C}$$

$$\frac{d\tilde{w}(s,\pi)}{d\theta}=\frac{T}{\mu}l^s \tag{6d}$$

$$\frac{d\tilde{w}(s,-\pi)}{d\theta}=\frac{Q}{\mu}h^s \tag{6e}$$

The solution of the ordinary differential equation (6a) is

$$\tilde{w}(s,\theta)=\begin{cases}A_1(s)\cos\theta s+B_1(s)\sin\theta s, & 0\le\theta\le\pi\\ A_2(s)\cos\theta s+B_2(s)\sin\theta s, & -\pi\le\theta\le 0\end{cases} \tag{7}$$

Where the coefficients $A_1(s),B_1(s),A_2(s)\ and\ B_2(s)$ are functions of s to be calculated using the transformed boundary conditions. (6a-6e)

3.0 The Wiener -Hopf equation

The emergence of two unknown functions $F(s)$ and $G(s)$ from the boundary conditions presupposes a Wiener-Hopf problem.

Now substituting the boundary condition eqn. (6c) into eqn. (7) gives

$$A_1(s)=b^2F(s)\ ,A_2(s)=b^2F(s) \tag{8a,b}$$

Taking the derivatives of eqn. (7) gives

$$\frac{d\tilde{w}(s,0^+)}{d\theta}=sB_1(s)\ ,\frac{d\tilde{w}(s,0^-)}{d\theta}=sB_2(s) \tag{8c,d}$$

Thus from eqn. (8c,d)

$$\frac{d\tilde{w}(s,0^+)}{d\theta}-\frac{d\tilde{w}(s,0^-)}{d\theta}=sB_1(s)\ -sB_2(s)=\frac{b^2}{\mu}G(s) \tag{8e}$$

Also further evaluation using the boundary conditions gives the following equations

$$\frac{d\tilde{w}(s,\pi)}{d\theta}=\frac{T}{\mu}l^s\ ,\ \frac{d\tilde{w}(s,-\pi)}{d\theta}=\frac{Q}{\mu}h^s \tag{9a}$$

$$B_1(s)=\frac{Tl^s+\mu sb^sF(s)\sin\pi s}{\mu s\cos\pi s} \tag{9b}$$

$$B_2(s)=\frac{Qh^s-\mu sb^sF(s)\sin\pi s}{\mu s\cos\pi s} \tag{9c}$$

Hence

$$\frac{b^s}{\mu s}G(s)=\frac{d\tilde{w}(s,0^+)}{d\theta}-\frac{d\tilde{w}(s,0^-)}{d\theta}=B_1(s)-B_2(s)$$

$$=\frac{Tl^s+\mu sb^sF(s)\sin\pi s}{\mu s\cos\pi s}-\left[\frac{Ql^s-\mu sb^sF(s)\sin\pi s}{\mu s\cos\pi s}\right]$$

$$=\frac{Tl^s-Qh^s}{\mu s\cos\pi s}+\frac{2\mu sb^sF(s)\sin\pi s}{\mu s\cos\pi s}$$

$$=\frac{b^s}{\mu s}\left[\frac{T\left(\frac{l}{b}\right)^s-Q\left(\frac{h}{b}\right)^s}{\cos\pi s}+\frac{2\mu sF(s)\sin\pi s}{\cos\pi s}\right]$$

Therefore

$$G(s)=\frac{T\left(\frac{l}{b}\right)^s-Q\left(\frac{h}{b}\right)^s}{\cos\pi s}+\frac{2\mu sF(s)\sin\pi s}{\cos\pi s} \tag{9d}$$

Let

$$D(s)=\frac{T}{\mu}\left(\frac{l}{b}\right)^s-\frac{Q}{\mu}\left(\frac{h}{b}\right)^s \tag{9e}$$

Thus

$$\frac{1}{\mu}G(s)=\frac{D(s)}{\cos\pi s}+\frac{2sF(s)\sin\pi s}{\cos\pi s} \tag{9f}$$

Simplifying further, we obtain the following Wiener Hopf equation.

$$F_-(s)=\frac{\cos\pi}{2s\sin\pi s}\left[\frac{1}{\mu}G_+(s)-\frac{D(s)}{\cos\pi s}\right]=N(s)\left[\frac{1}{\mu}G_+(s)-\frac{D(s)}{\cos\pi s}\right] \tag{10}$$

Let

$$N(s)=\frac{\cos\pi s}{2s\sin\pi s} \tag{11a}$$

Decomposing $N(s)$ into a quotient of the form

$$N(s)=\frac{N_+(s)}{N_-(s)} \tag{11b}$$

Where

$$N_+(s) = \frac{\prod_{k=1}^{\infty}\left(1+\frac{2s}{k}\right)e^{\psi s}}{\prod_{k=1}^{\infty}\left(1+\frac{s}{k}\right)\prod_{k=1}^{\infty}\left(1+\frac{s}{k}\right)} \tag{11c}$$

$$N_-(s) = \frac{2\pi s^2}{N_+(-s)} \tag{11d}$$

We have

$$F_-(s) = \frac{N_+(s)}{N_-(s)}\left[\frac{1}{\mu}G_+(s) - \frac{D(s)}{\cos\pi s}\right] \tag{12a}$$

This produces a separation of the form

$$F_-(s)N_-(s) = \frac{1}{\mu}N_+(s)G_+(s) - N_+(s)\frac{D(s)}{\cos\pi s} \tag{12b}$$

Decomposing $\frac{1}{\cos\pi s}$ into sum of functions, we have

$$\frac{1}{\cos\pi s} = H_-(s) + H_+(s) \tag{12c}$$

Where

$$H_-(s) = \frac{1}{\pi}\left\{\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{s-\alpha_k}\right\} \tag{12d}$$

$$H_-(s) = \frac{1}{\pi}\left\{\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{s+\alpha_k}\right\} \tag{12e}$$

Where

$$\alpha_k = \frac{2k-1}{2} \tag{12f}$$

Hence eqn. (8) becomes

$$F_-(s)N_-(s) = \frac{1}{\mu}N_+(s)G_+(s) - N_+(s)D(s)H_-(s) - N_+(s)D(s)H_+(s) \tag{13a}$$

Next , we decompose the mixed term $N_+(s)D(s)H_-(s)$ to sums one of which is a term with removable singularities. i.e.

$$N_+(s)D(s)H_-(s) = M_+(s) + M_-(s) \tag{13b}$$

Were

$$M_-(s) = \frac{1}{\pi}\left\{\sum_{k=1}^{\infty}\frac{(-1)^{k+1}D(\alpha_k)N_+(\alpha_k)}{s-\alpha_k}\right\} \tag{13c}$$

$$M_{+}(s)=\frac{1}{\pi}\left\{\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\left[D(s)N_{+}(s)-D(\alpha_k)N_{+}(\alpha_k)\right]}{s-\alpha_k}\right\} \tag{13d}$$

Consequently eqn.(13a) becomes

$$F_{-}(s)N_{-}(s)+M_{-}(s)=\frac{1}{\mu}N_{+}(s)G_{+}(s)-N_{+}(s)D(s)H_{+}(s)-M_{+}(s) \tag{13e}$$

Now because the functions analytic in the left half plane are equal to functions analytic in the right half plane, each function is an analytic continuation of the other with the fundamental strip as their strip of equality. Therefore, each side is bounded and analytic in the entire s -plane and by Sturm-Liouville's theorem must a constant.[8]
Hence eqn. (3e) becomes

$$F_{-}(s)N_{-}(s)+M_{-}(s)=\frac{1}{\mu}N_{+}(s)G_{+}(s)-N_{+}(s)D(s)H_{+}(s)-M_{+}(s)=c \tag{14}$$

Since eqn. (14) is true for all s, it must be true for $s=0$
Thus

$$F_{-}(0)N_{-}(0)+M_{-}(0)=c \tag{15}$$

Since

$$F_{-}(0)\neq 0\,,\ and\ N_{-}(0)=0$$
$$M_{-}(0)=c \tag{16a}$$

Therefore eqn.(14) becomes

$$F_{-}(s)=\frac{M_{-}(0)-M_{-}(s)}{N_{-}(s)} \tag{16b}$$

$$F_{-}(s)=\left(\frac{M_{-}(0)-M_{-}(s)}{s}\right)\frac{N_{+}(-s)}{s}$$

[Use has been made of eqn. (11d)] (16c)
Were

$$M_{-}(0)=\frac{1}{\pi}\left\{\sum_{k=1}^{\infty}\frac{(-1)^{k+1}D(\alpha_k)N_{+}(\alpha_k)}{(\alpha_k)}\right\} \tag{16d}$$

Now substituting the values of $A_1,\ A_2\ ,B_1,\ B_2$ into eqn. (7) and simplifying, we have

$$\tilde{w}(s,\theta)=\begin{cases}\dfrac{Tl^{s}\sin\theta s}{\mu s\cos\pi s}+\dfrac{b^{s}F(s)\cos(\pi-\theta)s}{\cos\pi s}, & 0\leq\theta\leq\pi\\[2ex] \dfrac{Qh^{s}\sin\theta s}{\mu s\cos\pi s}+\dfrac{b^{s}F(s)\cos(\pi+\theta)s}{\cos\pi s}, & -\pi\leq\theta\leq 0\end{cases} \tag{17}$$

Substituting eqn. (16b) into eqn. (17), we have

$$\tilde{w}(s,\theta)=\begin{cases}\dfrac{Tl^{s}\sin\theta s}{\mu s\cos\pi s}+\left(\dfrac{M_{-}(0)-M_{-}(s)}{N_{-}(s)}\right)\dfrac{b^{s}\cos(\pi-\theta)s}{\cos\pi s}, & 0\leq\theta\leq\pi\\[2ex] \dfrac{Qh^{s}\sin\theta s}{\mu s\cos\pi s}+\left(\dfrac{M_{-}(0)-M_{-}(s)}{N_{-}(s)}\right)\dfrac{b^{s}\cos(\pi+\theta)s}{\cos\pi s}, & -\pi\leq\theta\leq 0\end{cases} \tag{18}$$

Substituting eqn. (16c) into eqn. (18), we have

$$\tilde{w}(s,\theta)=\begin{cases}\dfrac{Tl^s\sin\theta s}{\mu s\cos\pi s}+\left(\dfrac{M_-(0)-M_-(s)}{s}\right)\dfrac{N_+(-s)}{s}\dfrac{b^s\cos(\pi-\theta)s}{\cos\pi s}, & 0\le\theta\le\pi\\[2ex] \dfrac{Qh^s\sin\theta s}{\mu s\cos\pi s}+\left(\dfrac{M_-(0)-M_-(s)}{s}\right)\dfrac{N_+(-s)}{s}\dfrac{b^s\cos(\pi+\theta)s}{\cos\pi s}, & -\pi\le\theta\le 0\end{cases}\tag{19}$$

The displacement $w(r,\theta)$ are given by

$$w(r,\theta)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{w}(s,\theta)r^{-s}dr\tag{20}$$

Using eqns. (20) ,(19)

$$w(r,\theta)=\begin{cases}\dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty}\left[\dfrac{T\left(\frac{l}{b}\right)^s\sin\theta s}{\mu s\cos\pi s}+\left(\dfrac{M_-(0)-M_-(s)}{s}\right)\dfrac{N_+(-s)}{s}\dfrac{b^s\cos(\pi-\theta)s}{\cos\pi s}\right]\left(\dfrac{r}{b}\right)^{-s}ds\ ,0\le\theta\le\pi\\[2ex] \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty}\left[\dfrac{Q\left(\frac{h}{b}\right)^s\sin\theta s}{\mu s\cos\pi s}+\left(\dfrac{M_-(0)-M_-(s)}{s}\right)\dfrac{N_+(-s)}{s}\dfrac{b^s\cos(\pi-\theta)s}{\cos\pi s}\right]\left(\dfrac{r}{b}\right)^{-s}ds,-\pi\le\theta\le 0\end{cases}\tag{21}$$

The Bromwich integral in eqn. (21) can be evaluated by Cauchy residue method, using Jordan's Lemma. Note also that $\dfrac{\sin\theta s}{s}$ and $\dfrac{M_-(0)-M_-(s)}{s}$ have removable singularities at $s=0$. The residue at $s=0$,may be ignored, because they lead to constants which do not alter the solution of a Neumann boundary value problem of the type we are solving. The singularities of $\dfrac{N_+(-s)}{s}$ and $\dfrac{1}{\cos\pi s}$ are simple poles located at $s=0$, for $\dfrac{N_+(-s)}{s}$ with residue $N_+(0)=0$, and at $s=\pm\left(\dfrac{2k-1}{2}\right), k=1,2,3,\ldots$ for $\dfrac{1}{\cos\pi s}$

It is not a difficult task to show the displacements

For $0\le\theta\le\pi,\quad r>b$

$$w(r,\theta)=-\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n\left[\frac{\frac{T}{\mu}\left(\frac{l}{b}\right)^{\frac{2n-1}{2}}}{2n-1}\sin\left(\frac{2n-1}{2}\right)\theta\right]+$$

$$2\left[\left(\frac{M_-(0)-M_-\left(\frac{2n-1}{2}\right)}{2n-1}\right)\frac{N_+\left(\frac{1-2n}{2}\right)}{2n-1}\cos(\pi-\theta)\left(\frac{2n-1}{2}\right)\right]\left(\frac{r}{b}\right)^{\frac{1}{2}(1-2n)}\tag{22a}$$

For $0\le\theta\le\pi,\quad 0<r<b$

$$w(r,\theta)=-\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n\left[\frac{\frac{T}{\mu}\left(\frac{l}{b}\right)^{\frac{1-2n}{2}}}{2n-1}\sin\frac{\theta}{2}(2n-1)\right]+$$

$$2\left[\left(\frac{M_-(0)-M_-\left(\frac{2n-1}{2}\right)}{2n-1}\right)\frac{N_+\left(\frac{1-2n}{2}\right)}{2n-1}\cos\frac{1}{2}(\pi-\theta)(2n-1)\right]\left(\frac{r}{b}\right)^{\frac{1}{2}(2n-1)} \qquad (22b)$$

For $-\pi\le\theta\le 0,\quad r>b$

$$w(r,\theta)=-\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n\left[\frac{-\frac{Q}{\mu}\left(\frac{h}{b}\right)^{\frac{1}{2}(1-2n)}}{2n-1}\sin\frac{\theta}{2}(1-2n)\right]+$$

$$2\left[\left(\frac{M_-(0)-M_-\left(\frac{2n-1}{2}\right)}{2n-1}\right)\frac{N_+\frac{1}{2}(2n-1)}{1-2n}\cos(\pi-\theta)\frac{1}{2}(1-2n)\right]\left(\frac{r}{b}\right)^{\frac{1}{2}(1-2n)} \qquad (22c)$$

For $-\pi\le\theta\le 0,\quad r<b$

$$w(r,\theta)=-\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n\left[\frac{-\frac{Q}{\mu}\left(\frac{h}{b}\right)^{\frac{1}{2}(1-2n)}}{2n-1}\sin\frac{\theta}{2}(1-2n)\right]+$$

$$2\left[\left(\frac{M_-(0)-M_-\left(\frac{1-2n}{2}\right)}{1-2n}\right)\frac{N_+\frac{1}{2}(2n-1)}{1-2n}\cos(\pi-\theta)\frac{1}{2}(1-2n)\right]\left(\frac{r}{b}\right)^{\frac{1}{2}(2n-1)} \qquad (22d)$$

4.0 RESULTS AND DISCUSSION

The displacements everywhere in the body has been obtained as given in eqn. (22a-d). To investigate the state of stress at the tips of the inhomogeneity, we need only the local displacements in the region defined by $0\le r\le b$, $-\pi\le\theta\le 0$, $0\le\theta\le\pi$ that is eqn. (22b, d). The inhomogeneity tip that is located at the end of the crack tip is approached as $r\to 0$. The displacement field corresponding to this case is represented asymptotically by the dominant terms obtained when $n=1$ and given as $r\to 0$ by

$$w(r,\theta)=\frac{2}{\pi}\begin{cases}-\left[\frac{T}{\mu}\left(\frac{l}{b}\right)^{\frac{1}{2}}\sin\frac{\theta}{2}+2\left[M_-(0)-M_-\left(\frac{1}{2}\right)\right]N_+\left(-\frac{1}{2}\right)\cos\frac{1}{2}(\pi-\theta)\right]\left(\frac{r}{b}\right)^{-\frac{1}{2}} & ,0\le\theta\le\pi\\ -\left[\frac{Q}{\mu}\left(\frac{h}{b}\right)^{\frac{1}{2}}\sin\frac{\theta}{2}+2\left[M_-(0)-M_-\left(\frac{1}{2}\right)\right]N_+\left(-\frac{1}{2}\right)\cos\frac{1}{2}(\pi+\theta)\right]\left(\frac{r}{b}\right)^{-\frac{1}{2}} & ,-\pi\le\theta\le 0\end{cases}$$

$$=\frac{2}{\pi}\begin{cases}-\left[\frac{T}{\mu}\left(\frac{l}{b}\right)^{\frac{1}{2}}\sin\frac{\theta}{2}+2\left[M_-(0)-M_-\left(\frac{1}{2}\right)\right]N_+\left(-\frac{1}{2}\right)\sin\frac{\theta}{2}\right]\left(\frac{r}{b}\right)^{-\frac{1}{2}} & ,0\le\theta\le\pi\\ -\left[\frac{Q}{\mu}\left(\frac{h}{b}\right)^{\frac{1}{2}}\sin\frac{\theta}{2}+2\left[M_-(0)-M_-\left(\frac{1}{2}\right)\right]N_+\left(-\frac{1}{2}\right)\sin\frac{\theta}{2}\right]\left(\frac{r}{b}\right)^{-\frac{1}{2}} & ,-\pi\le\theta\le 0\end{cases} \qquad (23)$$

$$\sigma_{rz}(r,\theta)=\frac{1}{\mu\pi\sqrt{b}}\begin{cases}-\left[\frac{T}{\mu}\left(\frac{l}{b}\right)^{-\frac{1}{2}}+2\left[M_-(0)-M_-\left(-\frac{1}{2}\right)\right]N_+\left(-\frac{1}{2}\right)\right]r^{-\frac{1}{2}}\sin\frac{\theta}{2} & ,0\le\theta\le\pi\\ \left[\frac{Q}{\mu}\left(\frac{h}{b}\right)^{-\frac{1}{2}}-2\left[M_-(0)-M_-\left(\frac{1}{2}\right)\right]N_+\left(-\frac{1}{2}\right)\right]\left(\frac{r}{b}\right)r^{\frac{1}{2}}\sin\frac{\theta}{2} & ,-\pi\le\theta\le 0\end{cases} \tag{24}$$

$$\sigma_{r\theta}(r,\theta)=\frac{1}{\mu\pi\sqrt{b}}\begin{cases}-\left[\frac{T}{\mu}\left(\frac{l}{b}\right)^{-\frac{1}{2}}+2\left[M_-(0)-M_-\left(-\frac{1}{2}\right)\right]N_+\left(\frac{1}{2}\right)\right]r^{-\frac{1}{2}}\cos\frac{\theta}{2} & ,0\le\theta\le\pi\\ \left[-\frac{Q}{\mu}\left(\frac{h}{b}\right)^{-\frac{1}{2}}-2\left[M_-(0)-M_-\left(\frac{1}{2}\right)\right]N_+\left(\frac{1}{2}\right)\right]\left(\frac{r}{b}\right)r^{\frac{1}{2}}\cos\frac{\theta}{2} & ,-\pi\le\theta\le 0\end{cases} \tag{25}$$

4.1 Stress field at the inner inhomogeneity tip

Since we are interested in the displacement $w(r,\theta)$ at the inner inhomogeneity tip, we use the local coordinate system (ρ,ϕ) as shown in Fig. 1. Noting that $\frac{r}{b}=\left(1+\frac{\rho}{b}\cos\phi\right)+0\left(\frac{\rho^2}{b^2}\right)$

Employing the methods introduced by choi and Earmme [9] we obtain the displacement as

$$w(\rho,\phi)=c_0-\frac{2}{\pi}M_-(0)\left(\frac{\rho}{b}\right)^{\frac{1}{2}}\cos\frac{\theta}{2} \tag{26}$$

Where c_0 is a constant representing a rigid body motion

The stresses at near the inner inhomogeneity tip are given as

$$\sigma_{\theta z}(\rho,\phi)=-\frac{1}{\pi}M_-(0)\left(\frac{\rho}{b}\right)^{\frac{1}{2}}\sin\frac{\theta}{2} \tag{27a}$$

$$\sigma_{\rho z}(\rho,\phi)=-\frac{1}{\rho\pi\sqrt{b}}M_-(0)^{\frac{1}{2}}\cos\frac{\theta}{2} \tag{27b}$$

4.2 Stress intensity factor at the infinite crack tip (or outer tip of the inhomogeneity) k_{III}^{outer}

At the infinite crack tip, it can be shown that the displacement and stress fields are as follows

$$w(r,\theta)=\frac{2}{\pi}\begin{cases}\left[\frac{T}{\mu}\left(\frac{l}{b}\right)^{\frac{1}{2}}\right]+\frac{2\sqrt{2}}{\pi\sqrt{\pi}}\left[\left(1-\left(\frac{l}{b}\right)^{\frac{1}{2}}\right)\frac{T}{\mu}-\left(1+\left(\frac{h}{b}\right)^{\frac{1}{2}}\frac{Q}{\mu}\right)\right]\left(\frac{r}{b}\right)^{\frac{1}{2}}\sin\frac{\theta}{2}, & 0\le\theta\le\pi\\ \left[\frac{Q}{\mu}\left(\frac{l}{b}\right)^{-\frac{1}{2}}\right]-\frac{2\sqrt{2}}{\pi\sqrt{\pi}}\left[\left(1-\left(\frac{l}{b}\right)^{\frac{1}{2}}\right)\frac{T}{\mu}-\left(1+\left(\frac{h}{b}\right)^{\frac{1}{2}}\frac{Q}{\mu}\right)\right]\left(\frac{r}{b}\right)^{\frac{1}{2}}\sin\frac{\theta}{2}, & -\pi\le\theta\le 0\end{cases} \tag{28a}$$

$$\sigma_{rz}(r,\theta)=\frac{1}{\mu\pi\sqrt{\pi}}\begin{cases}\left[\frac{T}{\mu}\left(\frac{l}{b}\right)^{\frac{1}{2}}\right]+\frac{2\sqrt{2}}{\pi\sqrt{\pi}}\left[\left(1-\left(\frac{l}{b}\right)^{\frac{1}{2}}\right)\frac{T}{\mu}-\left(1+\left(\frac{h}{b}\right)^{\frac{1}{2}}\frac{Q}{\mu}\right)\right]\left(\frac{r}{b}\right)^{-\frac{1}{2}}\sin\frac{\theta}{2}, & 0\le\theta\le\pi\\ \left[\frac{Q}{\mu}\left(\frac{h}{b}\right)^{-\frac{1}{2}}\right]-\frac{2\sqrt{2}}{\pi\sqrt{\pi}}\left[\left(1-\left(\frac{l}{b}\right)^{\frac{1}{2}}\right)\frac{T}{\mu}-\left(1+\left(\frac{h}{b}\right)^{\frac{1}{2}}\frac{Q}{\mu}\right)\right]\left(\frac{r}{b}\right)^{-\frac{1}{2}}\sin\frac{\theta}{2}, & -\pi\le\theta\le 0\end{cases} \tag{28b}$$

$$\sigma_{\theta z}(r,\theta)=\frac{1}{\mu\pi\sqrt{\pi}}\begin{cases}\left[\frac{T}{\mu}\left(\frac{l}{b}\right)^{\frac{1}{2}}\right]+\frac{2\sqrt{2}}{\pi\sqrt{\pi}}\left[\left(1-\left(\frac{l}{b}\right)^{\frac{1}{2}}\right)\frac{T}{\mu}-\left(1+\left(\frac{h}{b}\right)^{\frac{1}{2}}\frac{Q}{\mu}\right)\right]r^{\frac{1}{2}}\cos\frac{\theta}{2}, & 0\le\theta\le\pi\\ \left[\frac{Q}{\mu}\left(\frac{h}{b}\right)^{-\frac{1}{2}}\right]-\frac{2\sqrt{2}}{\pi\sqrt{\pi}}\left[\left(1-\left(\frac{l}{b}\right)^{\frac{1}{2}}\right)\frac{T}{\mu}-\left(1+\left(\frac{h}{b}\right)^{\frac{1}{2}}\frac{Q}{\mu}\right)\right]r^{-\frac{1}{2}}\cos\frac{\theta}{2}, & -\pi\le\theta\le 0\end{cases} \tag{28c}$$

The stress intensity factor at the outer inhomogeneity tip k_{III}^{outer} is given as

$$k_{III}^{outer}=\sqrt{\frac{2}{\pi b}}\begin{cases}+\left[T\left(\frac{l}{b}\right)^{\frac{1}{2}}\right]+\frac{2\sqrt{2}}{\pi\sqrt{\pi}}\left[\left(1-\left(\frac{l}{b}\right)^{\frac{1}{2}}\right)T-\left(1+\left(\frac{h}{b}\right)^{\frac{1}{2}}\right)Q\right], & 0\le\theta\le\pi\\ -\left[Q\left(\frac{h}{b}\right)^{-\frac{1}{2}}\right]-\frac{2\sqrt{2}}{\pi\sqrt{\pi}}\left[\left(1-\left(\frac{l}{b}\right)^{\frac{1}{2}}\right)T-\left(1+\left(\frac{h}{b}\right)^{\frac{1}{2}}\right)Q\right], & -\pi\le\theta\le 0\end{cases} \tag{29}$$

For self -equilibrated loading $Q=-T$ and $l=h$

$$k_{III}^{outer}=\sqrt{\frac{2}{\pi b}}\begin{cases}\left[T\left(\frac{l}{b}\right)^{\frac{1}{2}}+\frac{4\sqrt{2}}{\pi\sqrt{\pi}}\right], & 0\le\theta\le\pi\\ \left[T\left(\frac{l}{b}\right)^{-\frac{1}{2}}+\frac{4\sqrt{2}}{\pi\sqrt{\pi}}\right], & -\pi\le\theta\le 0\end{cases} \tag{30}$$

Denote by k_{III}^{ou}, the stress intensity factor for the case of $b=0$, the main crack without inhomogeneity, it is given by Tada and Paris [10] as

$$k_{III}^{ou}=\frac{T\sqrt{2}}{\sqrt{\pi l}} \tag{31}$$

For the case defined

$$k_{III}^{nor}=\frac{\left|k_{III}^{outer}\right|}{k_{III}^{ou}}=\frac{T\left|\left(\frac{l}{b}\right)^{\frac{1}{2}}+\frac{4\sqrt{2}}{\pi\sqrt{\pi}}\right|\sqrt{\frac{2}{\pi l}}}{T\sqrt{\frac{2}{\pi l}}}=\frac{l}{b}+\frac{4\sqrt{2}}{\pi^{\frac{3}{2}}}\left(\frac{l}{b}\right)^{\frac{1}{2}} \tag{32}$$

Similarly, the stress intensity factor at the inner tip of the inhomogeneity is obtained as

$$k_{III}^{inner}=\frac{2\sqrt{2}}{\sqrt{\pi b}}T\left(\frac{l}{b}\right)^{\frac{1}{2}}l \tag{33}$$

Conclusion

The existence of k_{III}^{outer} and k_{III}^{inner} implies that crack initiation can start either at the outer tip or at the inner tip depending on the strength of the stresses at the tip of inhomogeneity.

Let k_{III}^{0} denote the stress intensity factor at the crack tip for the case of an infinite plane with an infinite crack that goes beyond the origin with an extension of length b whose value was given by Choi and Earmme as

$$k_{III}^{ou}=\frac{T\sqrt{2}}{\sqrt{\pi l}} \tag{34}$$

Then we define the non-dimensional quantity $k_{III}^{normal}=k_{III}^{nor}$ where

$$k_{III}^{nor}=\frac{k_{III}^{inner}}{k_{III}^{o}}=2\frac{l}{b} \tag{35}$$

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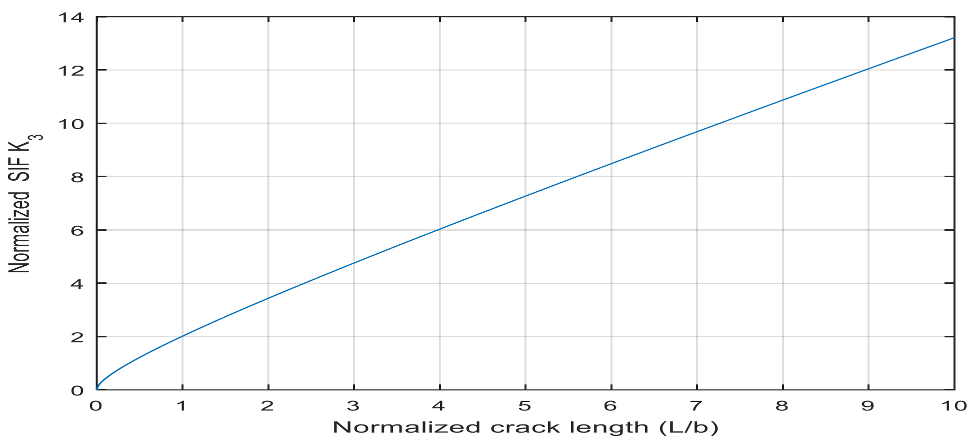

Fig.2. Normalized stress intensity factor at the outer tip of the inhomogeneity.

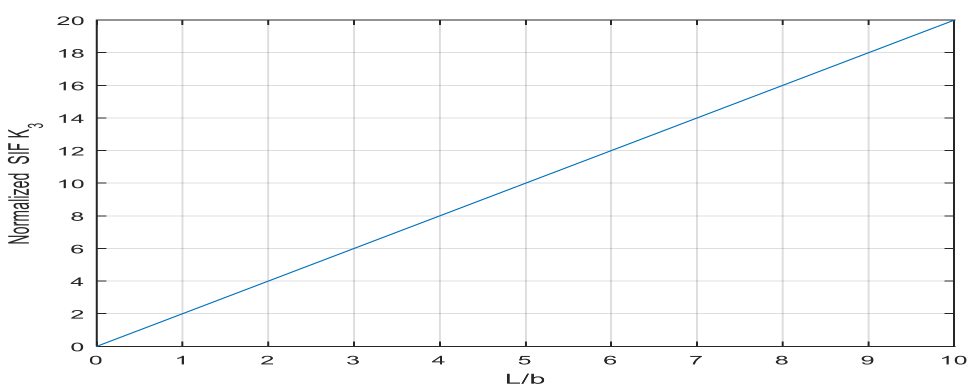

Fig.3. Normalized stress intensity factor at the inner tip of the inhomogeneity.