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LYAPUNOV UNIFORM ASYMPTOTIC STABILITY OF CAPUTO FRACTIONAL DYNAMIC EQUATIONS ON TIME SCALE USING A GENERALIZED DERIVATIVE

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ABSTRACT

In this work, we establish the uniform asymptotic stability using a generalized concept (herein referred to as Caputo fractional delta derivative and Caputo fractional delta Dini derivative of order $\alpha \in (0,1)$ for Caputo fractional derivatives on an arbitrary time domain T, which is a closed subset of R. Combining the continuous and discrete time domains, we create a unified framework for uniform asymptotic stability analysis on time scales. This work also incorporates an illustrative example to demonstrate the relevance, effectiveness, and applicability of the established stability results over that of the integer order.

1. Introduction

In recent years, the study of fractional calculus has gained significant attention due to its ability to capture complex dynamics and model real-world problems more accurately and efficiently. As a generalization of integer order derivatives and integrals, also referred to as differentiation and integration to an arbitrary order ([23]), fractional calculus has proven to be a powerful tool in understanding intricate systems. Building on this foundation, numerous studies have utilized the Lyapunov second method, also known as the Lyapunov direct method, with remarkable outcomes in comprehending the qualitative and quantitative characteristics of dynamical systems.

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One benefit of using the Lyapunov direct method is that it does not require knowledge of the solution to the differential equation under study ([25]). In [1, 2, 3, 5, 7], several types of fractional derivatives of Lyapunov functions used in stability investigations of differential equations, including Caputo fractional derivative, Dini fractional derivative, and Caputo fractional Dini derivative, were applied. However, the most preferred, as pointed out by the authors, is the Caputo Fractional derivative

$$ {}_{t_0}^{C}D_t^{\alpha}V\big(t,x(t)\big)=\frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}(t-s)^{-\alpha}\frac{d}{ds}(V(s,x(s)))ds,\qquad t\in[t_0,T) $$

because it is easier to handle and has a more realistic application. Still, the authors noted that the function $V\big(t,x(t)\big)$ needs to be continuously differentiable which poses another challenge. This disadvantage does not affect the other Lyapunov function derivatives, so the authors obtained sufficient conditions for these derivatives using a continuous Lyapunov function that does not need to be continuously differentiable. In [2], it was noted that the Dini fractional derivative

$$ D_+^{\alpha}V(t,x;t_0)=\lim_{h\to 0_+}\sup\frac{1}{h^{\alpha}}\left\{V(t,x)-\sum_{r=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^{r+1}\binom{\alpha}{r}V(t-rh,x-h^{\alpha}f(t,x))\right\} $$

where $V:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}_+$, is continuous, $f:[\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n]$, h is a positive number and $\binom{\alpha}{r}=\frac{\alpha(\alpha-1)\dots(\alpha-r+1)}{r!}$,

maintains the idea of fractional derivatives, since it depends not only on the present point (t), but also on the initial point (t_0). Yet, it does not depend on the initial state $V(t_0,x_0)$. So, a better definition

$$ {}_{t_0}^{C}D_+^{\alpha}V(t,x(t))=\lim_{h\to 0_+}\sup\frac{1}{h^{\alpha}}\left\{V(t,x(t))-V\big(t_0,x(t_0)\big)-\sum_{r=1}^{\left[\frac{t-t_0}{h}\right]}(-1)^{r+1}\binom{\alpha}{r}\left[V(t-rh,x(t)-h^{\alpha}f(t,x(t)))-V\big(t_0,x(t_0)\big)\right]\right\} \tag{1.1} $$

was considered more suitable. (see [1])

The Caputo fractional Dini derivative (1.1) has been utilized to examine various types of stability in Caputo fractional differential equations with continuous domains, as seen in [1, 4]. As explained in [19] and [8], a more holistic and practicable examination of stability can be achieved if it is conducted across different time domains. In [1, 2, 3, 4, 5], stability results were obtained for a continuous time, which ignores discrete details, while in [9, 18, 20], the domains considered are discrete, ignoring the continuous time domains. However, in practice, some systems undergo smooth and abrupt changes almost simultaneously, while others may have more than one time scale or frequency. Modeling such phenomena is more realistically represented as a dynamic system that includes continuous and discrete times, that is, time as an arbitrary closed subset of real numbers known as the time scale or measure chain, denoted by T [12]. Dynamic equations on time scales are defined on discrete, continuous (connected), or a combination of both, serving as a foundation for a broader analysis of difference and differential systems [14]. This work focuses on the Lyapunov uniform asymptotic stability analysis of Caputo fractional dynamic equations on time scales using a generalized definition for the delta derivative of a Lyapunov function introduced in our previous work [15]. The aim is to provide a unified and comprehensive understanding, extending the uniform asymptotic stability properties from the classical sense to the fractional-order sense.

Recently, several authors have explored fractional dynamic systems on time scales due to their promising advantages in modeling, mechanics, and population dynamics (see [28]). As highlighted in [10, 11, 16, 21, 22, 24], much of the existing literature primarily focuses on the existence and uniqueness of solutions to fractional dynamic equations on time scales, with the Caputo derivative garnering significant attention. The most recent work on Lyapunov stability analysis of fractional dynamic equations on time scales, presented in [15], concentrates on stability and asymptotic stability, underscoring the need for research on uniform asymptotic stability.

Building on the existence and uniqueness results for Caputo-type fractional dynamic equations on time scales established in [7], and the comparison and stability results in [15], we extend the uniform asymptotic stability result in [17] to fractional orders and generalize the Lyapunov uniform asymptotic stability results for Caputo fractional differential equations in [1] to a broader (unified) domain, namely, time scales. By establishing criteria for uniform asymptotic stability in Caputo fractional dynamic equations, this research not only expands classical Lyapunov uniform asymptotic stability analysis to fractional-order systems but also contributes to the recent literature [15].

The investigation unfolds by delving into basic definitions of important terminologies, and remarks that sets the stage for our contributions. In section 3, we reintroduce some lemmas as given in [15], which are important in establishing the main result in section 4. In section 5, we give a example to show the advantage and effectiveness of our result. This result contributes not only to the theoretical advancements in fractional calculus but also extend the result on integer-order dynamic equations on time scales to fractional order. To emphasize the relevance and effectiveness of the derived stability criteria, we present a detailed example, illustrating the importance and applicability of our results.

Preliminaries, Definitions, and Notations

The foundational principles of dynamic equations, encompassing derivatives and integrals, can be extended to non-integer orders by applying fractional calculus. This generalization to non-integer orders becomes particularly relevant when exploring dynamic equations on a time scale, allowing for a versatile and comprehensive analysis of system behavior across continuous and discrete time domains. In this section, we shall set the foundation, introduce notations, and give definitions that will be used in establishing the main results.

Definition 2.1. ([8]). For $t \in \mathbb{T}$, the forward jump operator $\sigma: \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T}: s > t\},$$

and the backward jump operator $\rho: \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T}: s < t\}.$$

The following conditions hold:

(i) If $\sigma(t) > t$, then t is termed right-scattered.

(ii) If $\rho(t) < t$, then t is termed left-scattered.

(iii) If $t < \max\mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense.

(iv) If $t > \min\mathbb{T}$ and $\rho(t) = t$, then t is called left-dense.

Definition 2.2. ([8]). The graininess function $\mu: \mathbb{T} \to [0, \infty)$ for $t \in \mathbb{T}$ is defined by

$$\mu(t) = \sigma(t) - t,$$

where $\sigma(t)$ is the forward jump operator.
The derivative is taken using the set $\mathbb{T}^k$, which is derived from the time scale $\mathbb{T}$ as follows:

- If T has a left-scattered maximum M, then $\mathbb{T}^k = \mathbb{T} \backslash \{M\}$.
- Otherwise, $\mathbb{T}^k = \mathbb{T}$. ([8]).

Definition 2.3. (Delta Derivative) ([8]). Let $h: \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. The delta derivative h^Δ also known as the Hilger derivative is defined as:

$$h^\Delta(t) = \frac{h(\sigma(t)) - h(s)}{\sigma(t) - s}, \qquad s \neq \sigma(t),$$

provided the limit exist.
If t is right dense, the delta derivative of $h: \mathbb{T} \to \mathbb{R}$, becomes

$$h^\Delta(t) = \lim_{s \to t} \frac{h(t) - h(s)}{(t) - s},$$

and if t is right scattered, the Delta derivative becomes

$$h^\Delta(t) = \frac{h^\sigma(t) - h(t)}{\mu(t)},$$

where $h^\sigma(t) = \sigma(t)$.

Definition 2.4 ([13])**.** A function $h: \mathbb{T} \to \mathbb{R}$ is called right-dense continuous if it is continuous at every right-dense point in $\mathbb{T}$ and has finite left-sided limits at left-dense points of $\mathbb{T}$. The set of all right-dense continuous functions is denoted by

$$C_{rd} = C_{rd}(\mathbb{T}).$$

Definition 2.5 ([13])**.** Let $[a, b]$ be a closed and bounded interval in $\mathbb{T}$. A function $H: [a, b] \to \mathbb{R}$ is called a delta antiderivative of a function $h: [a, b] \to \mathbb{R}$ if H is continuous on $[a, b]$, delta differentiable on $[a, b)$, and satisfies $H^\Delta(t) = h(t)$ for all $t \in [a, b)$. The delta integral is then defined as

$$\int_a^b h(t)\Delta t = H(b) - H(a), \qquad \forall a, b \in \mathbb{T}.$$

Remark 2.1 ([13])**.** All right dense continuous functions are delta integrable.

Remark 2.2. Let $a \leq b$,

$$[a, b]_\mathbb{T} = t \in \mathbb{T}: a \leq t \leq b,$$

$$(a, b)_\mathbb{T} = t \in \mathbb{T}: a < t < b,$$

$$[a, b)_\mathbb{T} = t \in \mathbb{T}: a \leq t < b,$$

$$(a, b]_\mathbb{T} = t \in \mathbb{T}: a < t \leq b,$$

are all intervals on a time scale $\mathbb{T}$.

Definition 2.6. ([8]). Let $a, b \in \mathbb{T}$ and $h \in C_{rd}$ then, integration on a time scale $\mathbb{T}$ is defined as follows:

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b h(t)\Delta t = \int_a^b h(t)dt$$

where $\int_a^b h(t)dt$ is the usual Riemann integral from calculus.

(ii) If $[a, b]_{\mathbb{T}}$ consists of only isolated points, then

$$\int_a^b h(t)\Delta t = \begin{cases} \sum_{t\in[a,b]} \mu(t)h(t) & \text{if } a < b \\ 0 & \text{if } a = b \\ -\sum_{t\in[a,b]} \mu(t)h(t) & \text{if } a > b \end{cases}$$

(iii) If there exists a point $\sigma(t) > t$, then

$$\int_t^{\sigma(t)} h(s)\Delta s = \mu(t)f(t).$$

Definition 2.7 ([13]). A function $\phi: [0, r] \to [0, \infty)$ is of class κ if it is continuousand strictly increasing on $[0, r]$ with $\phi(0) = 0$.

Definition 2.8 ([13]). A continuous function $\nu: \mathbb{R}^n \to \mathbb{R}$ with $\nu(0) = 0$ is called positive definite (negative definite) on the domain D if there exists a function $\phi \in \kappa$ such that $\phi(|\chi|) \leq \nu(\chi)(\phi(|\chi|) \leq -\nu(\chi))$ for $\chi \in D$.

Definition 2.9 ([13]). A continuous function $\nu: \mathbb{R}^n \to \mathbb{R}$ with $\nu(0) = 0$ is called positive semidefinite (negative semi-definite) on D if $\nu(\chi) \geq 0$ $(\nu(\chi) \leq 0)$ for all $\chi \in D$ and it can also vanish for some for some $\chi \neq 0$.

Definition 2.10 ([17]). Assume $V \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+]$, $h \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n]$ and $\mu(t)$ is the graininess function, then the derivative of $V(t, \chi)$ is defined as;

$$D_{-}V^{\Delta}(t,\chi) = \lim_{\mu(t)\to 0} \inf \frac{V(t,\chi) - V(t-\mu(t), \chi-\mu(t)h(t,\chi))}{\mu(t)} \tag{2.1}$$

$$D^{+}V^{\Delta}(t,\chi) = \lim_{\mu(t)\to 0} \sup \frac{V(t+\mu(t), \chi+\mu(t)h(t,\chi)) - V(t,\chi)}{\mu(t)} \tag{2.2}$$

If V is differentiable, then $D_{-}V^{\Delta}(t,\chi) = D^{+}V^{\Delta}(t,\chi) = V^{\Delta}(t,\chi)$.

Definition 2.11 (Fractional integral on Time Scale) ([7]). Let $\alpha \in (0,1)$, $[a, b]$ be an interval on $\mathbb{T}$ and h an integrable function on $[a, b]$. Then the fractional integral of order α of h is defined by

$${}_{a}^{\mathbb{T}}I_t^{\alpha} h^{\Delta}(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\Delta s.$$

Definition 2.12. (Caputo Derivative on Time Scale) ([7]). Let $\mathbb{T}$ be a time scale, $t \in \mathbb{T}$, $0 < \alpha < 1$, and $h: \mathbb{T} \to \mathbb{R}$. The Caputo fractional derivative of order α of h is defined by

$$ {}_{a}^{\mathbb{T}}D_{t}^{\alpha}h^{\Delta}(t) = \frac{1}{\Gamma(1-\alpha)}\int_{a}^{t}(t-s)^{-\alpha}h^{\Delta^{n}}(s)\Delta s. $$

Definition 2.13. Let $h \in C_{rd}^{\alpha}[\mathbb{T}, \mathbb{R}^{n}]$, the Grunwald-Letnikov fractional delta derivative is given by

$$ {}^{GL\mathbb{T}}D_{0}^{\alpha}h^{\Delta}(t) = \lim_{\mu\to 0^{+}}\frac{1}{\mu^{\alpha}}\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\alpha}C_{r}[h(\sigma(t)-r\mu)] \quad t \geq t_0 \tag{2.3} $$

and the the Grunwald-Letnikov fractional delta dini derivative is given by :

$$ {}^{GL\mathbb{T}}D_{0^{+}}^{\alpha}h^{\Delta}(t) = \lim_{\mu\to 0^{+}}\sup\frac{1}{\mu^{\alpha}}\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\alpha}C_{r}[h(\sigma(t)-r\mu)] \quad t \geq t_0 \tag{2.4} $$

where $0 < \alpha < 1$, ${}^{\alpha}C_{r} = \frac{q(q-1)\cdots(q-r+1)}{r!}$, and $\left[\frac{(t-t_0)}{\mu}\right]$ denotes the integer part of the function $\frac{(t-t_0)}{\mu}$.

Observe that if the domain is $\mathbb{R}$, then (2.1) becomes

$$ {}^{GL\mathbb{T}}D_{0^{+}}^{\alpha}h^{\Delta}(t) = \lim_{d\to 0^{+}}\sup\frac{1}{d^{\alpha}}\sum_{r=0}^{[\frac{(t-t_0)}{d}]}(-1)^{r\alpha}C_{r}[h(t-rd)] \quad t \geq t_0 $$

Remark 2.3. It is necessary to note that the relationship between the Caputo fractional delta derivative and the Grunwald-Letnikov fractional delta derivative is given by

$$ {}^{C\mathbb{T}}D_{0^{+}}^{\alpha}h^{\Delta}(t) = {}^{GL\mathbb{T}}D_{0^{+}}^{\alpha}[h(t)-h(t_0)]^{\Delta} $$

Substituting (2.3) into (2.5) we have that the Caputo fractional delta derivative becomes

$$ {}^{C\mathbb{T}}D_{0}^{\alpha}h^{\Delta}(t) = \lim_{\mu\to 0^{+}}\frac{1}{\mu^{\alpha}}\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\alpha}C_{r}[h(\sigma(t)-r\mu)-h(t_0)] \quad t \geq t_0 $$

$$ {}^{C\mathbb{T}}D_{0}^{\alpha}h^{\Delta}(t) = \lim_{\mu\to 0^{+}}\frac{1}{\mu^{\alpha}}\left\{h\big(\sigma(t)\big)-h(t_0)+\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\alpha}C_{r}[h(\sigma(t)-r\mu)-h(t_0)]\right\} \tag{2.6} $$

And the Caputo fractional delta Dini derivative becomes

$$ {}^{C\mathbb{T}}D_{0^{+}}^{\alpha}h^{\Delta}(t) = \lim_{\mu\to 0^{+}}\sup\frac{1}{\mu^{\alpha}}\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\alpha}C_{r}[h(\sigma(t)-r\mu)-h(t_0)] \quad t \geq t_0 \tag{2.7} $$

which is equivalent to

$$ {}^{C\mathbb{T}}D_{0}^{\alpha}h^{\Delta}(t) = \lim_{\mu\to 0^{+}}\sup\frac{1}{\mu^{\alpha}}\left\{h\big(\sigma(t)\big)-h(t_0)+\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\alpha}C_{r}[h(\sigma(t)-r\mu)-h(t_0)]\right\}, \quad t > t_0 \tag{2.8} $$

For notation simplicity, we shall represent the Caputo fractional derivative of order α as ${}^{C\mathbb{T}}D^{\alpha}$ and the Caputo fractional delta dini derivative of order α as ${}^{C\mathbb{T}}D_{+}^{\alpha}$.

Definition 2.14. The trivial solution $x = 0$ of (3.1) is uniformly asymptotically stable if it is uniformly stable and locally attractive, that is there exists a $\delta_0 > 0$ such that $||x(t_0)|| = \delta_0$ implies $\lim_{t\to\infty} ||x(t_0)|| = 0$ for $t_0, t \in \mathbb{T}$.

Now, we introduce the derivative of the Lyapunov function using the Caputo fractional delta Dini derivative of $h(t)$ given in (2.7).

Definition 2.15. We define the Caputo fractional delta Dini derivative of the Lyapunov function $V(t,x) \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+]$ (which is locally Lipschitzian with respect to its second argument and $V(t,0) \equiv 0$) along the trajectories of solutions of the system (3.1) as:

$$^{C\mathbb{T}}D_+^{\alpha}V^{\Delta}(t,x) = \lim_{\mu\to 0^+} \sup \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^r ({}^{\alpha}C_r)[V(\sigma(t) - r\mu, x(\sigma(t)) - \mu^{\alpha} f\big(t, x(t)\big) - V(t_0, x_0)]$$

And can be expanded as

$$.^{C\mathbb{T}}D_+^{\alpha}V^{\Delta}(t,x) = \lim_{\mu\to 0^+} \sup \frac{1}{\mu^{\alpha}} \Big\{ V(\sigma(t), x(\sigma(t)) - V(t_0, x_0) + \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r+1} ({}^{\alpha}C_r)[V(\sigma(t) - r\mu, x(\sigma(t)) - \mu^{\alpha} f(t, x(t) - V(t_0, x\_0)] \Big\} \tag{2.9}$$

Where $t \in \mathbb{T}, x, x_0 \in \mathbb{R}^n$, $\mu = \sigma(t) - t$ and $x(\sigma(t) - \mu^{\alpha} f(t,x) \in \mathbb{R}^n$.

If $\mathbb{T}$ is discrete and $V\,(t, x(t))$ is continuous at t, the Caputo fractional delta Dini derivative of the Lyapunov function in discrete times, is given by:

$$.^{C\mathbb{T}}D_+^{\alpha}V^{\Delta}(t,x) = \frac{1}{\mu^{\alpha}} [\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r+1} ({}^{\alpha}C_r)[V(\sigma(t), x(\sigma(t)) - V(t_0, x_0))] \tag{2.10}$$

and if $\mathbb{T}$ is continuous, that is $\mathbb{T} = \mathbb{R}$, and $V(t, x(t))$ is continuous at t, we have that

$$.^{C\mathbb{T}}D_+^{\alpha}V^{\Delta}(t,x) = \lim_{d\to 0^+} \sup \frac{1}{d^{\alpha}} \Big\{ V(t, x(t)) - V(t_0, x_0) - \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r+1} ({}^{\alpha}C_r)[V(t - rd, x(t)) - d^{\alpha} f(t, x(t) - V(t_0, x\_0)] \Big\} \tag{2.11}$$

Notice that (2.11) is the same in [1] where $d > 0$

Given that $\lim_{N\to\infty} \sum_{r=0}^{N} (-1)^r {}^{\alpha}C_r = 0$ where $\alpha \in (0,1)$, and $\lim_{\mu\to 0^+} [\frac{(t-t_0)}{\mu}] = \infty$, then it is easy to see that

$$\lim_{\mu\to 0^+} \sum_{r=0}^{N} (-1)^r {}^{\alpha}C_r = -1 \tag{2.12}$$

Also from (2.7) and since the Caputo and Riemann-Liouville formulations coincide when $h(t_0) = 0$, ([1]) then we have that

$$^{C\mathbb{T}}D_+^{\alpha}h^{\Delta}(t) = {}^{RL\mathbb{T}}D_+^{\alpha}h^{\Delta}(t) = \lim_{\mu\to 0^+} \sup \frac{1}{\mu^{\alpha}} [\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r+1} ({}^{\alpha}C_r)[h(\sigma(t) - r\mu)], \; t \geq t_0 \tag{2.13}$$

So that

$$\lim_{\mu\to 0^+} \sup \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^r {}^{\alpha}C_r = {}^{RL\mathbb{T}}D^{\alpha}(1) = \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}, \; t \geq t_0. \tag{2.14}$$

INEQUALITIES ON FRACTIONAL DYNAMIC EQUATIONS ON TIME SCALE AND COMPARISON RESULTS.

For the purpose of this work, we consider the Caputo fractional dynamic system of order α with $0 < \alpha < 1$

$$ {}^{C\mathbb{T}}D^{\alpha}x^{\Delta} = f(t,x), \ t \in \mathbb{T}, \quad x(t_0) = x_0, \ t_0 \geq 0 \tag{3.1}$$

where $f \in C_{rd}[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}^N]$, $f(t,0) \equiv 0$ and ${}^{C\mathbb{T}}D^{\alpha}x^{\Delta}$ is the Caputo fractional delta derivative of $x \in \mathbb{R}^N$ of order α with respect to $t \in \mathbb{T}$. Let $x(t) = x(t,t_0,x_0) \in C_{rd}^{\alpha}[\mathbb{T}, \mathbb{R}^N]$ (the fractional derivative of order alpha of $x(t)$ exist and it is rd-continuous) be a solution of (3.1) and assume the solution exists and is unique (results on existence and uniqueness of (3.1) is contained in [7]), this work aims to investigate the uniform asymptotic stability of the system (3.1).

To do this, we shall use the Caputo fractional dynamic system of the form

$$ {}^{C\mathbb{T}}D^{\alpha}u^{\Delta} = g(t,u), \ u(t_0) = u_0 > 0 \tag{3.2}$$

where $u \in \mathbb{R}_+$, $g: \mathbb{T} \times \mathbb{R}_+ \to \mathbb{R}_+$ and $g(t,0) \equiv 0$. (3.2) is called the comparison system. For this work, we will assume that the function $g \in [\mathbb{T} \times \mathbb{R}_+ \to \mathbb{R}_+]$, is such that for any initial data $(t_0, u_0) \in \mathbb{T} \times \mathbb{R}_+$, the system (3.2) with $u(t_0) = u_0$ has a unique solution $u(t) = u(t;t_0,u_0) \in C_{rd}^{\alpha}[\mathbb{T}, \mathbb{R}_+]$ see [7].

Definition 3.1. The zero solution of (3.1) is said to be:

(S_1) Uniformly stable if for every $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exist $\delta = \delta(\epsilon) > 0$ such that for any $x_0 \in \mathbb{R}^N$, the inequality $||x_0|| < \delta$ implies $x(t;t_0,x_0) < \epsilon$ for $t \geq t_0$

(S_1) Uniformly asymptotically stable if it uniformly stable and there exist numbers $\delta_0 = \delta_0(\epsilon)$ and $T = T(\epsilon)$ such that for $t \geq t_0 + T$, the inequality $||x\_0|| \leq \delta$ implies $x(t;t\_0,x\_0) < \epsilon$.

Lemma 3.1. ([15]) Assume that

(i) $g \in C_{rd}[\mathbb{T} \times \mathbb{R}_+]$ and $g(t,u)\mu$ is non-decreasing in u.

(ii) $V \in C_{rd}[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}_+]$ be locally Lipschitz in the second variable such that

$$ {}^{C\mathbb{T}}D^{\alpha}V^{\Delta}(t,x) = g\big(t, V(t,x)\big), (t,x) \in \mathbb{T} \times \mathbb{R}^N \tag{3.3}$$

(iii) $z(t) = z(t;t_0,u_0)$ is the maximal solution of (3.2) existing on $\mathbb{T}$.

Then

$$ V\big(t, x(t)\big) \leq z(t), \ t \geq t_0 \tag{3.4}$$

provided that

$$ V(t_0, x_0) \leq u_0 \tag{3.5}$$

where $x(t) = x(t;t_0x_0)$ is any solution of (3.1), $t \in \mathbb{T}$ $t \geq t_0$

Lemma 3.2 (Uniform Stability)**.** [[15]] Assume the following

(1) $g \in C_{rd}[\mathbb{T} \times \mathbb{R}_+, \mathbb{R}_+]$ and $g(t,u)$ is non-decreasing in u with $g(t,u) \equiv 0$.

(2) $V(t,x(t)) \in C[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}_+]$ be such that

(i) V is locally Lipschitz in x with $V(t,0) \equiv 0$

(ii) $b(||x||) \leq V(t,x) \leq a(||x||)$ where $a, b \in K$

(iii) For any points $t, t_0 \in \mathbb{T}$ and $x, x_0 \in \mathbb{R}^N$, the inequality

$${}^{C\mathbb{T}}D_{+}^{\alpha}V^{\Delta}(t,x(t)) \leq g(t,V(t,x(t)))$$

holds.

(3) The zero solution of FrDE (3.2) is uniformly stable Then the zero solution of the FrDE (3.1) is uniformly stable.

Main Results

In this section, we will obtain sufficient conditions for the uniform asymptotic stability of the system (2.9)

Theorem 4.1 (Uniform asymptotic Stability)**.** Assume the following

(1) $V(t,x(t)) \in C[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}_+]$ and locally Lipschitz in x such that $V(t,0) \equiv 0$.

(2) $g \in C_{rd}[\mathbb{T} \times \mathbb{R}_+, \mathbb{R}_+]$ and $g(t,u)$ is non-decreasing in u with $g(t,u) \equiv 0$.

(3) For $(t,x) \in \mathbb{T} \times \mathbb{R}^N$,

$${}^{C\mathbb{T}}D_{+}^{\alpha}V^{\Delta}(t,x) \leq -c(\|x\|)$$

where $c \in K$.

4. $b(||x||) \leq V(t,x) \leq a(||x||)$ where $a, b \in K$.

Then the zero solution $x = 0$ of the FrDE (3.1) is uniformly asymptotically stable.

Proof. Let $x^*(t)$ be any solution of (3.1). Then $x^* = 0$ is uniformly asymptotically stable, if it is uniformly stable and uniformly attractive. By Theorem 3.2, $x^* = 0$ is uniformly stable.
We now show that $x^* = 0$ is uniformly attractive. That is, for any $\eta > 0$, there exist $T = T(\eta) > 0$ such that for any $t_0 \in \mathbb{T}$, $x_0 \in \mathbb{R}^N$ with $\|x_0\| \leq \delta$, the inequality $\|x^*(t; t_0, x_0)\| < \eta$ holds for $t \geq t_0 + T$. Let $\lambda \in (0,\delta)$ be a constant such that

$$a(\lambda) < b(\delta), \quad \text{with} \quad \|x_0\| < \lambda \tag{4.1}$$

Combining condition (4) and (4.1) at (t_0, x_0), we have that

$$b(\|x_0\|) \leq V\,(t_0, x_0) \leq a(\|x_0\|) < a(\lambda) < b(\delta) \tag{4.2}$$

We can see clearly from (4.2) that $\| x_0 \| < \delta$.

Let $\eta = \eta(\epsilon)$, $\eta \in (0, \epsilon)$.

We claim that

$$\|x_0\| < \delta \Longrightarrow \|x^*(t)\| < \eta, \text{for} \quad t \geq t_0 + T \tag{4.3}$$

Assume (4.3) is not true, then there exists at least one $t \geq t_0 + T$, such that

$$\|x_0\| < \delta \Longrightarrow \|x^*(t)\| \geq \eta \tag{4.4}$$

Since $c \in K$, condition (2) of the theorem can be written as

$${}^{C\mathbb{T}}D_{+}^{\alpha}V^{\Delta}(t,x^*) \leq -c(\eta) \tag{4.5}$$

which is equivalent to the Volterra integral equation

$$V(t,x^*(t)) \leq V(t_0,x_0) - \frac{c(\eta)}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\,\Delta s \qquad \text{for} \quad t \geq t_0 + T \tag{4.6}$$

if $\mathbb{T} = \mathbb{R}$, then (4.6) becomes

$$V(t, x^*(t)) \leq V(t_0, x_0) - \frac{c(\eta)(t-t_0)}{\alpha\Gamma(\alpha)} \qquad (4.7)$$

and for sufficiently large t, (4.7) contradicts the condition (4) of the Theorem so (4.3) holds when $\mathbb{T} = \mathbb{R}$.
If $\mathbb{T}$ consists of only isolated points, then (4.6) becomes

$$V(t, x^*(t)) \leq V(t_0, x_0) - \frac{c(\eta)}{\Gamma(\alpha)} \sum_{j=1}^{i} \mu(t_j) r(t_j) \qquad (4.8)$$

for all $t_i, t_j \in \mathbb{T}$ and $r(t) = (t-s)^{\alpha-1}$. As $i \to \infty$, the right-hand side of (4.8) tends to $-\infty$ which is also a contradiction.

If T consists of t_i such that $\sigma(t) > t$, then (4.6) becomes

$$V(t, x^*(t)) \leq V(t_0, x_0) - \frac{c(\eta)}{\Gamma(\alpha)} \mu(t_i) r(t_i) \qquad (4.9)$$

where $r(t) = (t-s)^{\alpha-1}$. As $i \to \infty$, the right-hand side of (4.9) approaches $-\infty$. Which is another contradiction, implying that (4.3) holds across different time scales.

Therefore, the zero solution of (3.1) is uniformly asymptotically stable. □

and for sufficiently large t, (4.7) contradicts the condition (4) of the Theorem and in-fact definition of $V(t, x(t))$ and so (4.3) holds when $T = \mathbb{R}$.

If T consists of only isolated points, then (4.6) becomes

$$V\big(t, x^*(t)\big) \leq V(t_0, x_0) - \frac{c(\eta)}{\Gamma(\alpha)} \sum_{j=1}^{i} \mu\big(t_j\big) r(t_j) \qquad (4.8)$$

for all $t_i, t_j \in T$ and $r(t) = (t-s)^{\alpha-1}$, as $i \to \infty$, the right-hand side of (4.8) tends to $-\infty$ which is also a contradiction.

If T consists of t_i such that $\sigma(t) > t$, then (4.6) becomes

$$V\big(t, x^*(t)\big) \leq V(t_0, x_0) - \frac{c(\eta)}{\Gamma(\alpha)} \mu(t_i) r(t_i) \qquad (4.9)$$

Where $r(t) = (t-s)^{\alpha-1}$. As $i \to \infty$, the right-hand side of (4.9) approaches $-\infty$. Which is another contradiction, implying that (4.3) holds across different time scales.

Therefore, the zero solution of (3.1) is uniformly asymptotically stable.

Application

Consider the system of dynamic equations

$$^{CT}D^{\alpha}\chi_1(t) = g_1(t)\chi_1 - g_2(t)\chi_2$$
$$^{CT}D^{\alpha}\chi_1(t) = g_1(t)\chi_2 + g_2(t)\chi_1 \qquad (5.1)$$

For $t \geq t_0$, with initial conditions

$$\chi_1(t_0) = \chi_{10} \text{ and } \chi_2(t_0) = \chi_{20}$$

where $\chi = (\chi_1, \chi_2)\mathbb{R}^2$ and $g = (g_1, g_2) \epsilon \mathbb{R}^2$

Consider $V(t, \chi_1, \chi_2) = \chi_1^2 + \chi_2^2$, for $t \epsilon T$, $(\chi_1, \chi_2) \epsilon \mathbb{R}^2$ and choose $\alpha = 1$, so that (5.1) becomes an integer (first) order system. Then we compute the delta Dini derivative of

$V(t,\chi_1,\chi_2) = \chi_1^2 + \chi_2^2$ along the solution path of (5.1) as follows:

From (2.2) we have that

$$D^+V^\Delta(t,\chi) = \lim_{\mu(t)\to 0} \sup \frac{V(t+\mu(t),\chi+\mu(t)f(t,\chi)) - V(t,\chi)}{\mu(t)}$$

$$= \lim_{\mu(t)\to 0} \sup \frac{(\chi_1+\mu(t)f_1(t,\chi_1,\chi_2))^2+(\chi_2+\mu(t)f_2(t,\chi_1,\chi_2))^2-[\chi_1^2+\chi_2^2]}{\mu(t)}$$

$$= \lim_{\mu(t)\to 0} \sup \frac{(\chi_1^2+2\chi_1\mu(t)f_1+\mu^2(t)f_1^2+\chi_2^2+2\chi_2\mu(t)f_2+\mu^2(t)f_2^2-[\chi_1^2+\chi_2^2]}{\mu(t)}$$

$$= \lim_{\mu(t)\to 0} \sup \frac{2\chi_1\mu(t)f_1+\mu^2(t)f_1^2+2\chi_2\mu(t)f_2+\mu^2(t)f_2^2}{\mu(t)}$$

$$\leq 2\chi_1 f_1 + 2\chi_2 f_2$$

$$= 2\chi_1(g_1(t)\chi_1 - g_2(t)\chi_2) + 2\chi_2(g_1(t)\chi_2 + g_2(t)\chi_1)$$

$$= 2[g_1(t)\chi_1^2 - g_2(t)\chi_1\chi_2] + 2[g_1(t)\chi_2^2 + g_2(t)\chi_1\chi_2]$$

$$= 2g_1(t)[\chi_1^2 + \chi_2^2]$$

$$D^+V^\Delta(t,\chi) \leq 2g(t, V(t,\chi_1,\chi_2))$$

Now consider the consider the comparison equation

$$D^+u^\Delta = 2u > 0, \qquad u(0) = u_0 \tag{5.2}$$

Even though conditions (i)-(iii) of [17] are satisfied that is $V\epsilon C_{rd}[T\times\mathbb{R}^n, \mathbb{R}+]$,

$D^+V^\Delta(t,\chi) \leq g(t,V(t,\chi))$ and $\sqrt{\chi_1^2+\chi_2^2} \leq \chi_1^2+\chi_2^2 \leq 2(\chi_1^2+\chi_2^2)$, for $b\left||\chi|\right| = r$ and $a\left(\left||\chi|\right|\right) = 2r^2$, it is obvious to see that the solution of the comparison system (5.2) is not uniformly asymptotically stable, so we cannot deduce the uniform asymptotic stability properties of the system (5.1) by applying the basic definition of the Dini-derivative of a Lyapunov function of dynamic equation on time scale to the Lyapunov function

$$V(t,\chi_1,\chi_2) = \chi_1^2 + \chi_2^2.$$

Let us consider (5.1) with $\alpha\epsilon(0,1)$ and apply the new definition (2.9).

For $V(t,\chi_1,\chi_2) = \chi_1^2 + \chi_2^2$, for $t\epsilon T$ and $(\chi_1,\chi_2)\epsilon\mathbb{R}^2$. Then condition 1 of Theorem (4.1) is satisfied, for $b(\left||\chi|\right|) \leq V(t,\chi) \leq a(\left||\chi|\right|)$, with $b(r) = r$, $a(r) = 2r^2$, $a, b \in K$, so that the associated norm $\left||\chi|\right| = \sqrt{\chi_1^2+\chi_2^2}$.

Since

$$V(t,\chi_1,\chi_2) = \chi_1^2 + \chi_2^2$$

then $\sqrt{\chi_1^2+\chi_2^2} \leq \chi_1^2+\chi_2^2 \leq 2(\chi_1^2+\chi_2^2)$. From (2.9), we compute the Caputo fractional Dini derivative for $V(t,\chi_1,\chi_2) = \chi_1^2 + \chi_2^2$ as follows

$$ {}^{C\mathbb{T}}_{\cdot}D^{\alpha}_{+}V^{\Delta}(t,\chi) = \lim_{\mu\to 0^+}\sup\frac{1}{\mu^{\alpha}}\left\{V\left(\sigma(t),\chi(\sigma(t)) - V(t_0,\chi_0)\right)\right. $$

$$ \left. - \sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]} (-1)^{r+1}\,({}^{\alpha}C_r)\left[V\left(\sigma(t)-r\mu,\chi(\sigma(t)) - \mu^{\alpha}f(t,\chi(t))\right) - V(t_0,\chi_0)\right]\right\} $$

$$ = \lim_{\mu\to 0^+}\sup\frac{1}{\mu^{\alpha}}\{[(\chi_1(\sigma(t)))^2 + (\chi_2^2(\sigma(t)))^2] - [(\chi_{10})^2 + (\chi_{20})^2] $$

$$ + \sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]} (-1)^{r}\,({}^{\alpha}C_r)[(\chi_1(\sigma(t)) - \mu^{\alpha}f_1(t,\chi_1,\chi_2))^2 + (\chi_2(\sigma(t)) $$

$$ - \mu^{\alpha}f_2(t,\chi_1,\chi_2))^2((\chi_{10})^2 + (\chi_{20})^2)]\} $$

$$ = \lim_{\mu\to 0^+}\sup\frac{1}{\mu^{\alpha}}\{[(\chi_1(\sigma(t)))^2 + (\chi_2^2(\sigma(t)))^2] - [(\chi_{10})^2 + (\chi_{20})^2] $$

$$ + \sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]} (-1)^{r}\,({}^{\alpha}C_r)[(\chi_1(\sigma(t)))^2 - 2\chi_1(\sigma(t))\mu^{\alpha}f_1(t,\chi_1,\chi_2) + \mu^{2\alpha}(f_1(t,\chi_1,\chi_2))^2 $$

$$ + (\chi_2(\sigma(t)))^2 - 2\chi_2(\sigma(t))\mu^{\alpha}f_2(t,\chi_1,\chi_2) $$

$$ + \mu^{2\alpha}(f_2(t,\chi_1,\chi_2))^2 - ((\chi_{10})^2 + (\chi_{20})^2)]\} $$

$$ = \lim_{\mu\to 0^+}\sup\frac{1}{\mu^{\alpha}}\{- \sum_{r=0}^{\left[\frac{t-t_0}{\mu}\right]} (-1)^{r}\,({}^{\alpha}C_r)[(\chi_{10})^2 + (\chi_{20})^2] + \sum_{r=0}^{\left[\frac{t-t_0}{\mu}\right]} (-1)^{r}\,({}^{\alpha}C_r)[(\chi_1(\sigma(t)))^2 $$

$$ + (\chi_2(\sigma(t)))^2] $$

$$ - \sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]} (-1)^{r}\,({}^{\alpha}C_r\left[2\chi_1(\sigma(t))\mu^{\alpha}f_1(t,\chi_1,\chi_2) + 2\chi_2(\sigma(t))\mu^{\alpha}f_2(t,\chi_1,\chi_2)\right] $$

$$ + \sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]} (-1)^{r}\,({}^{\alpha}C_r)[\mu^{2\alpha}(f_1(t,\chi_1,\chi_2))^2 + \mu^{2\alpha}(f_2(t,\chi_1,\chi_2))^2]\} $$

$$= -\limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha}\{\sum_{r=0}^{\left[\frac{t-t_0}{\mu}\right]} (-1)^r\, ({}^{\alpha}C_r)[(\chi_{10})^2 + (\chi_{20})^2]\}$$

$$+ \limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha}\{\sum_{r=0}^{\left[\frac{t-t_0}{\mu}\right]} (-1)^r\, ({}^{\alpha}C_r)[\left(\chi_1(\sigma(t))\right)^2 + \left(\chi_2(\sigma(t))\right)^2]\}$$

$$- \limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha}\{\sum_{r=0}^{\left[\frac{t-t_0}{\mu}\right]} (-1)^r\, ({}^{\alpha}C_r)[2\chi_1(\sigma(t))\mu^\alpha f_1(t,\chi_1,\chi_2)$$
$$+ 2\chi_2(\sigma(t))\mu^\alpha f_2(t,\chi_1,\chi_2)]\}$$

Applying (2.12) and (2.14) we have

$$= -\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}((\chi_{10})^2 + (\chi_{20})^2 + \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}((\chi_1(\sigma(t)))^2 + \left(\chi_2(\sigma(t))\right)^2)$$
$$- [2\chi_1(\sigma(t))f_1(t,\chi_1,\chi_2) + 2\chi_2(\sigma(t))f_2(t,\chi_1,\chi_2)]$$
$$\le \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}((\chi_1(\sigma(t)))^2 + \left(\chi_2(\sigma(t))\right)^2) - [2\chi_1(\sigma(t))f_1(t,\chi_1,\chi_2)$$
$$+ 2\chi_2(\sigma(t))f_2(t,\chi_1,\chi_2)]$$

As $t \to \infty$, $\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}((\chi_1(\sigma(t)))^2 + \left(\chi_2(\sigma(t))\right)^2) \to 0$, then

$${}^{C\mathbb{T}}_{\cdot}D^{\alpha}_{+}V^{\Delta}(t,\chi_1,\chi_2) \le 2[\chi_1(\sigma(t))f_1(t,\chi_1,\chi_2) + \chi_2(\sigma(t))f_2(t,\chi_1,\chi_2)]$$

applying

$\chi(\sigma(t)) \le \mu^{CT}D^\alpha\chi(t) + \chi(t)$

$$= -2[\mu(t)f_1^2(t,\chi_1,\chi_2) + \chi_1(t)f_1(t,\chi_1,\chi_2) + \mu(t)f_2^2(t,\chi_1,\chi_2) + \chi_2(t)f_2(t,\chi_1,\chi_2)]$$

$$= -2[g_1(t)\chi_1^2 + g_2(t)\chi_2^2] - 2\mu(t)[(g_1(t)\chi_1 - g_2(t)\chi_2)^2 + (g_1(t)\chi_2 + g_2(t)\chi_1)^2] \qquad (5.3)$$

If $\mathbb{T} = \mathbb{R}$ we have that $\mu = 0$, so that (5.3) becomes;

$$= -2g_1(t)[\chi_1^2 + \chi_2^2]$$

Therefore,

$${}^{C\mathbb{T}}_{\cdot}D^{\alpha}_{+}V^{\Delta}(t,\chi_1,\chi_2) \le -2g(t,V(t,\chi_1,\chi_2))$$

Consider the comparison system

(5.4) $${}^{C\mathbb{T}}_{\cdot}D^{\alpha}_{+}u^{\Delta} = g(t,u) \le -2u$$

$${}^{C\mathbb{T}}_{\cdot}D^{\alpha}_{+}u^{\Delta} + 2u = 0$$

Applying the Laplace transform method, we obtain

(5.5) $$u(t) = u_0E_{\alpha,1}(-2t^{\alpha}), \quad \text{for } \alpha \in (0,1).$$

Now, let $u_0 < \delta$, then from (5.5), we have $\lim_{t\to\infty} u(t) = \lim_{t\to\infty} u_0E_{\alpha,1}(-2t^\alpha) = 0$ whenever

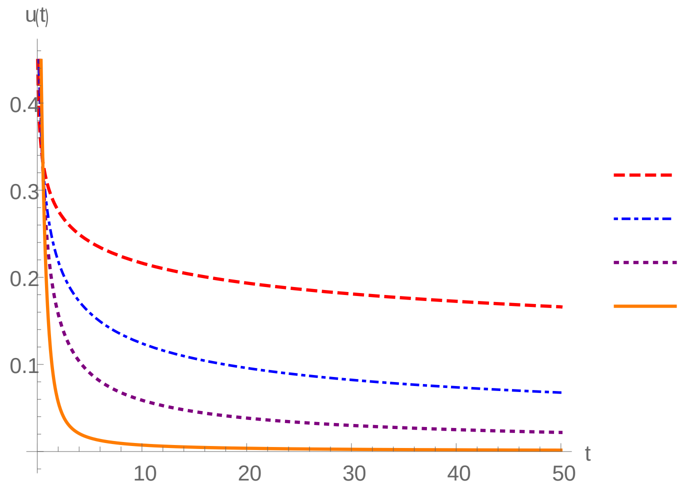
$$u_0 < \delta = \frac{\epsilon}{2E_{\alpha,1}}$$

If $\mathbb{T} = \mathbb{N}_0$ we have that $\mu = 1$, so that (5.3) becomes;

$$= -2[g_1(t)\chi_1^2 + g_2(t)\chi_2^2] - 2[(g_1(t)\chi_1 - g_2(t)\chi_2)^2 + (g_1(t)\chi_2 + g_2(t)\chi_1)^2]$$

$$^{C\mathbb{T}}_{\cdot}D_{+}^{\alpha}V^{\Delta}(t;\chi_1,\chi_2) \leq -2g_1(t)[\chi_1^2 + \chi_2^2]$$

we can also consider same comparison system as (5.4) leading to the same conclusion as (5.5) Since all the conditions of Theorem 4.1 are satisfied, and trivial solution of the comparison system (5.4) is uniformly asymptotically stable, then we conclude that the trivial solution of system (5.1) is uniformly asymptotically stable.



α= 0.2 α= 0.4 α= 0.6 α= 0.9

Figure 1. Graph of $u(t) = E_{\alpha,1}(-2t^{\alpha})$ against t

Figure 1 above shows the behaviour of $u(t)$ with respect to time for different values of α. It is obvious to see that $u(t) \to 0$ as $t \to \infty$.

CONCLUSION

In this paper, we have introduced a generalized approach to establishing uniform asymptotic stability criteria for Caputo fractional dynamic equations on arbitrary time domains. By establishing uniform asymptotic stability criteria based on the Caputo fractional delta derivative and Caputo fractional delta Dini derivative, we have created a unified framework for uniform asymptotic stability analysis on time scales. This framework effectively bridges the gap between continuous and discrete time domains, providing a powerful tool for predicting long-term stable behavior in complex dynamic systems.The significance of this research lies in its potential to ensure reliable and consistent outcomes in various applications, including control theory, signal processing, and engineering. By providing a unified framework for asymptotic stability analysis, we have paved the way for further research into the long-term stability of fractional dynamic systems. Our results demonstrate the effectiveness and applicability of the established uniform asymptotic stability criteria, showcasing their relevance in understanding and predicting the behavior of complex systems over time. We have also shown the theoretical applicability of this definition in Theorem 4.1 and its practical effectiveness in system (5.1).

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