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DEVELOPMENT AND APPLICATION OF MULTI-DERIVATIVE HYBRID BLOCK METHODS FOR SOLVING NONLINEAR FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper, two step implicit hybrid block multistep method, incorporating multi-derivatives is considered. The first incorporated second, third and fourth derivative, while the second incorporated second, third, fourth derivative and fifth derivative. The schemes are generated for the numerical solution of non-linear dynamical first order ordinary differential equations. The study made use of Bhaskara cosine approximation formula to generate hybrid points for the optimization of the numerical schemes generated by using collocation and interpolation technique. Power series is used as the basis function in approximating the solution. The methods are self-starting, of higher order, zero-stable, consistent and are A-stable. The methods are used to solve problems from chaos theory, SIR-model and multi-dimension problem to demonstrate the effectiveness of the method and its ability provide reliable solutions over existing methods.

1. Introduction

In this paper, we considered the method to approximate the solution of the first order initial value problem of the form:

$$y' = f(x, y), y(x_0) = y_0 \qquad (1.1)$$

Where: x_n, is the initial point, y_n is the solution at x_n. f is continuous within the interval of integration.

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Equation (1.1) is of interest to researchers because of its wide application in engineering control theory, chaos theory, dynamical system, non-linear system and other real-life problem, hence the study of the methods of its solution.

Numerical methods for Ordinary Differential Equations (ODEs) are essential tools in scientific computation, widely used to solve real-life problems. The rise of nonlinear problems has accompanied advancements in science and technology. Unlike linear problems, nonlinear problems cannot be tackled using standard linear techniques. Consequently, a new approach has been developed over the years to understand complex systems. Nonlinear dynamic systems often display complex behaviors such as population growth and chaos. While these systems can be used to examine chaotic or disordered phenomena and uncover intricate laws, this is not the sole focus of nonlinear problems [1,2]. To gain insights into a system's characteristics, it is crucial to construct an appropriate nonlinear mathematical model that accurately represents the underlying laws of the data. Nonlinear dynamics are diverse and vary in a more complicated manner based on the prior state. This complexity presents a significant challenge in practical engineering. Generally, finding an analytical solution is nearly impossible when a complex chaotic state is present. Therefore, when describing an unknown system state, people tend to prioritize methods that offer high approximation accuracy and ease of use over attempting to solve the exact problem.

Over the years, many approaches have been developed to solve the numerical solutions of nonlinear dynamic systems. The major methods include the perturbation method [3], averaging method [4], Runge-Kutta method [5], Euler method [6], gradient method [7], linear multistep method, among others. While these methods have certain benefits for handling specific systems, they often produce unsatisfactory results when applied to generic nonlinear dynamic systems. Issues such as reduced accuracy, increased complexity, large computational requirements, and Runge phenomena are common. Therefore, the current challenge is to find an effective approach for studying nonlinear dynamic systems that offers both high approximation accuracy and avoids the Runge phenomenon [8-14].

Considering the above discussion, we introduce two more direct methods for solving first-order nonlinear ODEs, applicable to special, stiff, nonlinear, and general forms of these equations. The proposed methods are two-step multi-derivative hybrid methods that utilizes four generated Bhaskara points as hybrid points to optimize the methods. These methods aim to be time-efficient, provide a wider integration range, and be economically reliable. The objective of this study is to develop two optimized multi-derivative methods with intra-step points for solving nonlinear dynamical systems.

The method is implemented has an implicit two-step method using Bhaskara points as hybrid points [15]. The order, zero stability, convergence and consistency of the method were studied. Some numerical problem which are non-linear will be solved and compared to others in literature to show the efficiency and accuracy.

1.1 Development of the methods.

The power series of the form below is considered in deriving the methods:

$$y(x)=\sum_{j=0}^{I+MC} \alpha_j x^j \qquad (2.1)$$

Where I is the interpolation point, C is number of collocation points for each derivative assuming all the collocations points at each derivative are the same, M is the number of derivatives.

$$\left.\begin{aligned} y(x_n) &= y_n \\ y'(x_n+v) &= f_{n+v} \\ y''(x_n+v) &= g_{n+v} \\ y'''(x_n+v) &= p_{n+v} \\ y^{iv}(x_n+v) &= \varpi_{n+v} \\ y^{v}(x_n+v) &= H_{n+v} \end{aligned}\right\} \tag{2.2}$$

$$v=\frac{5}{37},\frac{1}{2},1,\frac{3}{2},\frac{69}{37},2$$

The first derivative, second derivative, third derivative, fourth derivative and fifth derivative of the methods are:

$$y'(x)=\sum_{j=1}^{I+MC} j\alpha_j x^{j-1} \tag{2.3}$$

$$y''(x)=\sum_{j=2}^{I+MC} j(j-1)\alpha_j x^{j-2} \tag{2.4}$$

$$y'''(x)=\sum_{j=3}^{I+MC} j(j-1)(j-2)\alpha_j x^{j-3} \tag{2.5}$$

$$y^{iv}(x)=\sum_{j=4}^{I+MC} j(j-1)(j-2)(j-3)\alpha_j x^{j-4} \tag{2.6}$$

$$y^{v}(x)=\sum_{j=5}^{I+MC} j(j-1)(j-2)(j-3)(j-4)\alpha_j x^{j-5} \tag{2.7}$$

2.1 Derivation of the Fourth Derivative Method (FDM):

(2.1) is interpolated at $x=x_{n+1}$ while (2.3), (2.4), (2.5) and (2.6) are collocated at

$$x=x_{n+1},\, i=0,\frac{5}{37},\frac{1}{2},1,\frac{3}{2},\frac{69}{37},2 \tag{2.8}$$

The points i are the Bhaskara hybrid points that optimized the method, n represents the number of iterations for a step number of 2.

The system of equation gotten from (2.1), (2.3), (2.4), (2.5) and (2.6) can be written in the form below

$$y_w=D\psi_w \tag{2.9}$$

Where:

$$y_w=\begin{pmatrix} y_n, f_n, f_{n+\frac{5}{37}}, f_{n+\frac{1}{2}}, f_{n+1}, f_{n+\frac{3}{2}}, f_{n+\frac{69}{37}}, f_{n+2}, g_n, g_{n+\frac{5}{37}}, g_{n+\frac{1}{2}}, g_{n+1}, g_{n+\frac{3}{2}}, g_{n+\frac{69}{37}}, g_{n+2}, \\ p_n, p_{n+\frac{5}{37}}, p_{n+\frac{1}{2}}, p_{n+1}, p_{n+\frac{3}{2}}, p_{n+\frac{69}{37}}, p_{n+2}, \varpi_n, \varpi_{n+\frac{5}{37}}, \varpi_{n+\frac{1}{2}}, \varpi_{n+1}, \varpi_{n+\frac{3}{2}}, \varpi_{n+\frac{69}{37}}, \varpi_{n+2} \end{pmatrix}^T$$

And

$$\phi_w=\begin{pmatrix} \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}, \\ \alpha_{16}, \alpha_{17}, \alpha_{18}, \alpha_{19}, \alpha_{20}, \alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{24}, \alpha_{25}, \alpha_{26}, \alpha_{27}, \alpha_{28} \end{pmatrix}^T$$

The matrix D for (FDM) is shown in Appendix

Equation (2.9) is solved by matrix inversion technique which yield the continuous coefficients, which are then substituted into (2.1) to obtain its equivalent continuous scheme:

$$y(x)=y_n+h\left(\beta_0(x)f_n+\beta_{\frac{5}{37}}(x)f_{n+\frac{5}{37}}+\beta_{\frac{1}{2}}(x)f_{n+\frac{1}{2}}+\beta_1(x)f_{n+1}+\beta_{\frac{3}{2}}(x)f_{n+\frac{3}{2}}+\beta_{\frac{69}{37}}(x)f_{n+\frac{69}{37}}+\beta_2(x)f_2\right)+h^2\left(\gamma_0(x)g_n+\gamma_{\frac{5}{37}}(x)g_n\right.$$

(2.10)

Evaluating (2.10) at $x=0,\frac{5}{37},\frac{1}{2},\frac{3}{2},\frac{69}{37},2$gives the discrete schemes *which* form the block for the two-step fourth derivative block hybrid-points method with $m=4$.

2.2 Derivation of the Fifth Derivative Method (FIDM):

(2.1) is interpolated at $x=x_{n+1}$ while (2.3), (2.4), (2.5), (2.6) and (2.7) are collocated at

$$x=x_{n+1}, i=0,\frac{5}{37},\frac{1}{2},1,\frac{3}{2},\frac{69}{37},2 \qquad (2.11)$$

The points i are the Bhaskara hybrid points that optimized the method, n represents the number of iterations for a step number of 2.

The system of equation gotten from (2.1), (2.3), (2.4), (2.5) and (2.6) can be written in the form below

$$y_w=D\psi_w \qquad (2.12)$$

Where:

$$y_w=\begin{pmatrix} y_n, f_n, f_{n+\frac{5}{37}}, f_{n+\frac{1}{2}}, f_{n+1}, f_{n+\frac{3}{2}}, f_{n+\frac{69}{37}}, f_{n+2}, g_n, g_{n+\frac{5}{37}}, g_{n+\frac{1}{2}}, g_{n+1}, g_{n+\frac{3}{2}}, g_{n+\frac{69}{37}}, g_{n+2}, \\ p_n, p_{n+\frac{5}{37}}, p_{n+\frac{1}{2}}, p_{n+1}, p_{n+\frac{3}{2}}, p_{n+\frac{69}{37}}, p_{n+2}, \varpi_n, \varpi_{n+\frac{5}{37}}, \varpi_{n+\frac{1}{2}}, \varpi_{n+1}, \varpi_{n+\frac{3}{2}}, \varpi_{n+\frac{69}{37}}, \varpi_{n+2}, \\ H_n, H_{n+\frac{5}{37}}, H_{n+\frac{1}{2}}, H_{n+1}, H_{n+\frac{3}{2}}, H_{n+\frac{69}{37}}, H_{n+2} \end{pmatrix}^T$$

And

$$\phi_w=\begin{pmatrix} \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}, \\ \alpha_{16}, \alpha_{17}, \alpha_{18}, \alpha_{19}, \alpha_{20}, \alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{24}, \alpha_{25}, \alpha_{26}, \alpha_{27}, \alpha_{28}, \alpha_{29} \\ \alpha_{30}, \alpha_{31}, \alpha_{32}, \alpha_{32}, \alpha_{33}, \alpha_{34}, \alpha_{35} \end{pmatrix}^T$$

The matrix D for (FIDM) is shown in Appendix

Equation (2.12) is solved by matrix inversion technique which yield the continuous coefficients, which are then substituted into (2.1) to obtain its equivalent continuous scheme:

$$y(x)=y_n+h\left(\beta_0(x)f_n+\beta_{\frac{5}{37}}(x)f_{n+\frac{5}{37}}+\beta_{\frac{1}{2}}(x)f_{n+\frac{1}{2}}+\beta_1(x)f_{n+1}+\beta_{\frac{3}{2}}(x)f_{n+\frac{3}{2}}+\beta_{\frac{69}{37}}(x)f_{n+\frac{69}{37}}+\beta_2(x)f_2\right)+h^2\left(\gamma_0(x)g_n+\gamma_{\frac{5}{37}}(x)g_n\right. \quad (2.13)$$

Evaluating (2.13) at $x=0,\frac{5}{37},\frac{1}{2},\frac{3}{2},\frac{69}{37},2$gives the discrete schemes *which* form the block for the two-step fifth derivative block hybrid-points method with $m=4$.

2.3 Local Truncation Error:

2.3.1 Local Truncation Error of the Fourth Derivative Method:

The derived two-step methods are presented in this section. The linear operator is considered as:

$$L[y(x_n);h]=\sum_i\left[\alpha_i y(x_n+ih)-h\gamma_i f(x_n+ih)-h^2\beta_i g(x_n+ih)-h^3 k_i p(x_n+ih)-h^4\omega_i\varpi(x_n+ih)\right] \quad (2.14)$$

The function $y(x)$ is an arbitrary test function that is continuously differentiable in the interval [a, b]. Expanding $y(x_n+ih)$, $f(x_n+ih)$ and $g(x_n+ih)$ in Taylors series about x_n and factoring the coefficients of h to get

$$L[y(x_n);h]=C_0 y(x_n)+C_1 hy(x_n)+C_2h^2y^2(x_n)+C_3h^3y^3(x_n)+C_4h^4y^4(x_n)+\ldots+C_ph^py^p(x_n)+\ldots(2.15)$$

Where $c_i, i=0,1,2,\ldots$ are vectors.

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(2.16)

A method is of order p if $C_1=C_2=C_3=C_4\ldots=C_p=0, C_{p+1}\neq 0$. The C_{p+1} is the error constant and $C_{p+1}h^{p+1}y^{p+1}(x_n)$ is the principal truncation error at the point x_n. The local truncation error and order analysis of the block methods are found below.

From $Eq.(1.28)$, It was obtained for (1.15) that $C_1=C_2=\ldots=C_{28}=0$ and

$$C_{29}=\begin{pmatrix} \frac{-116372283146502977820187073177166391803904}{18927264030913067951579709278585770072479193743962596223671751979470062382812 5} \\ \frac{-2248093146594674787119}{444798579129829641766430970843124737075691168727040000000 0} \\ \frac{-9148763359953713}{148467543311539053552866782946675928093425664000000000} \\ \frac{2248093146594674787119}{444798579129829641766430970843124737075691168727040000000 0} \\ \frac{116372283146502977820187073177166391803904}{18927264030913067951579709278585770072479193743962596223671751979470062382812 5} \\ \frac{9148763359953713}{148467543311539053552866782946675928093425664000000000} \end{pmatrix}$$

(2.17)

The details of $C_1=C_2=\ldots=C_{28}=0$ is given in the appendix.

Therefore, the proposed method when $m=4$ Incorporating second, third and fourth is of order 28.

2.3.2 Local Truncation Error of the Fifth Derivative Method:

The derived two-step methods are presented in this section. The linear operator is considered as:

$$L\left[y(x_n);h\right]=\sum_i\left[\alpha_i y(x_n+ih)-h\gamma_i f(x_n+ih)-h^2\beta_i g(x_n+ih)-h^3 k_i p(x_n+ih)-h^4\omega_i\varpi(x_n+ih)-h^5 Y_i H(x_n+ih)\right]$$

(2.18)

The function $y(x)$ is an arbitrary test function that is continuously differentiable in the interval [a, b]. Expanding $y(x_n+ih)$, $f(x_n+ih)$ and $g(x_n+ih)$ in Taylors series about x_n and factoring the coefficients of h to get

$$L\left[y(x_n);h\right]=C_0 y(x_n)+C_1 hy(x_n)+C_2h^2y^2(x_n)+C_3h^3y^3(x_n)+C_4h^4y^4(x_n)++C_5h^5y^5(x_n)\ldots+C_ph^py^p(x_n)+\ldots(2.19)$$

Where $c_i, i=0,1,2,\ldots$ are vectors.

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A method is of order p if $C_1=C_2=C_3=C_4=C_5\ldots=C_p=0, C_{p+1}\neq 0$. The C_{p+1} is the error constant and $C_{p+1}h^{p+1}y^{p+1}(x_n)$ is the principal truncation error at the point x_n. The local truncation error and order analysis of the block methods are found below.

From $Eq.(1.32)$, It was obtained for (1.23) that $C_1=C_2=\ldots=C_{35}=0$ and

$$C_{36}=\left(\begin{array}{c} \frac{-2052874940526625404600601028459304834826 24}{6006511503033372473552041329743281577644285775559675585907566906268531303681714238578} \\ \frac{--48619450420178380036 4047}{1195300727586986831829597229956932470094877383035473011280130867200000000} \\ \frac{-243746251580479}{712454752675883073705194729540903848466204037091417915392000000} \\ \frac{-486194504201783800364047}{1195300727586986831829597229956932470094877383035473011280130867200000000} \\ \frac{-20528749405266254046006010284593048348262 4}{6006511503033372473552041329743281577644285775559675585907566906268531303681714238578} \\ \frac{-243746251580479}{712454752675883073705194729540903848466204037091417915392000000} \end{array}\right)$$

(2.21)

The details of $C_1=C_2=\ldots=C_{35}=0$ is given in the appendix.
Therefore, the proposed method when $m=5$ Incorporating second, third and fourth is of order 35.

2.4 Zero Stability:

2.4.1 Zero Stability of Fourth Derivative Method:

The two-step second, third and fourth derivative block methods can generally be written as a matrix difference equation as follows

$$A^{(1)}Y_w=A^{(0)}Y_{w-1}+h\left[B^{(0)}F_{w-1}+B^{(1)}F_w\right]+h^2\left[C^{(0)}G_{w-1}+C^{(1)}G_w\right]+h^3\left[E^{(0)}I_{w-1}+E^{(1)}I\right]+h^4\left[J^{(0)}L_{w-1}+J^{(1)}L\right] \quad (2.22)$$

And the matrices $A^{(1)},A^{(0)},B^{(1)},B^{(0)},C^{(1)},C^{(0)},E^{(0)}\ E^{(1)},J^{(0)}$ and $J^{(1)}$ are matrices whose entries are given by the coefficients of the methods, whose first characteristic polynomial is given as

$$\rho(\lambda)=\left|\lambda A^{(1)}-A^0\right| \quad (2.23)$$

Definition (Zero-stability): The block method (1.37) is said to be zero stable if the roots of the first characteristic polynomial $\rho(\lambda)$ satisfies $|\lambda_j|\leq 1, j=1,2,3,\ldots$ and for those roots with $|\lambda_j|=1$, the multiplicity must not exceed 1.

Using (1.38), we have

$$\rho(\lambda)=-\lambda^3(\lambda+1)=0 \quad \lambda=\{0,0,0,0,0,-1\} \quad (2.24)$$

Therefore, the method is zero stable since is satisfies $|\lambda_j|\leq 1$

2.4.2 Zero Stability for Fifth Derivative Method:

The two-step second third and fourth derivative block methods can generally be written as a matrix difference equation as follows

$$A^{(1)}Y_w=A^{(0)}Y_{w-1}+h\left[B^{(0)}F_{w-1}+B^{(1)}F_w\right]+h^2\left[C^{(0)}G_{w-1}+C^{(1)}G_w\right]+h^3\left[E^{(0)}I_{w-1}+E^{(1)}I\right]+h^4\left[J^{(0)}L_{w-1}+J^{(1)}L\right]+h^5\left[Q^{(0)}W_{w-} \quad (2.25)$$

And the matrices $A^{(1)},A^{(0)},B^{(1)},B^{(0)},C^{(1)},C^{(0)},E^{(0)}\ E^{(1)},J^{(0)}$ and $J^{(1)}$ are matrices whose entries are given by the coefficients of the methods, whose first characteristic polynomial is given as

$$\rho(\lambda)=\left|\lambda A^{(1)}-A^{0}\right| \tag{2.26}$$

Definition (Zero-stability): The block method (1.40) is said to be zero stable the roots of the first characteristic polynomial $\rho(\lambda)$ satisfies $|\lambda_j|\leq 1, j=1,2,3,\ldots$ and for those roots with $|\lambda_j|=1$, the multiplicity must not exceed 1.

Using (3.38), we have

$$\rho(\lambda)=-\lambda^3(\lambda+1)=0\ \lambda=\{0,0,0,0,0,-1\} \tag{2.27}$$

Therefore, the method is zero stable since is satisfies $|\lambda_j|\leq 1$

2.5 Consistency:

A linear multistep method is said to be consistent if the order of accuracy $p>1$. Therefore, we can infer from section 2.3.1 and section 2.3.2 that the two block methods are consistent.

2.6 Convergence

The necessary and sufficient condition for linear multistep method to be convergent is for it to be consistent and zero stable (Lambert, 1973). Following this theorem, each of the block methods developed are convergent.

2.7 Region of Absolute Stability:

Following Akinfenwa *et al* [16] the region of absolute stability is determined by obtaining the stability polynomial of the form:

$$\sigma(z)=\left(A^{(1)}-zB^{(1)}-z^2C^{(1)}\right)^{-1}\left(A^{(0)}+zB^{(0)}+z^2C^{(0)}\right) \tag{2.28}$$

Where $z=\lambda h$

The matrix $\sigma(z)$ has eigenvalues $\{0,0,0,\ldots,\lambda_k\}$, and the dominant eigenvalue $\lambda_k: C\rightarrow C$ is a rational function (called the stability function) with real coefficients given by

$$\lambda_k=\frac{P(z)}{P(-z)} \tag{2.29}$$

The stability functions show that for $\Re(z)<0, |\lambda_k|\leq 1$.

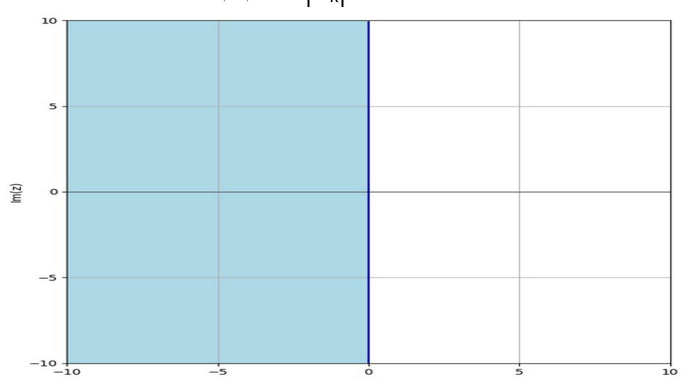

Figure 1 shows the stability region for the two stability polynomials and found to be an A-stable method since its region of absolute stability contains the left half-plane $C^{-¿¿}$.

3. Numerical Experiments

First-order dynamical systems are applied in various fields such as population dynamics, chemical equations, and vibration theory. They are implemented at this point. The resulting iterative methods are discretized into the following form:

$$y'_{n+j}=f(x_{n+j},y_{n+j}),\ y''_{n+j}=g(x'_{n+j},f_{n+j}),\ j=0,\frac{5}{37},\frac{1}{2},1,\frac{3}{2},\frac{29}{37},2 \tag{3.1}$$

And implemented as a block, it needs no starting values nor predictors. Using the known initial condition, $y(x_n)$ for $n=0,1,\ldots,N-2$, the first order IVPs are solved in the N mon-overlapping block points $[x_0,x_n]\ldots,[x_{N-2},x_n]$,with the step size defined in the usual way as $h=x_{n+1}-x_n$.

3.1 One dimensional non-linear Equations

Problem 1: The Mathieu equation is essential in the study of periodic linear vibrations or oscillations in various physical systems, including mechanical and electrical systems, crystal physics, quantum mechanics, and more. It arises when examining the behavior of systems subject to periodic forces or exhibiting periodic structures.

The Mathieu equation expressed as a system of two first order equations:

$$y_1' = y_2 \qquad (3.2)$$

$$y_2' = -(\delta + \epsilon \cos 2t)\, y_1 \qquad (3.3)$$

The problem is subjected to initial conditions $x_0 = 0.1,\ y_0 = 0$.

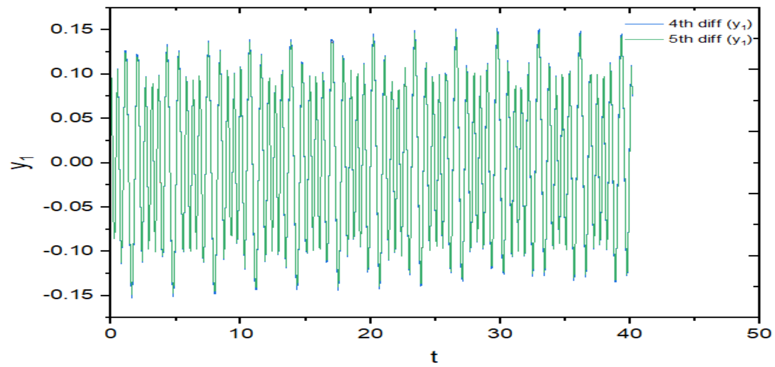

Fig 2: Comparison of Numerical Solutions for y_1 of Problem 1

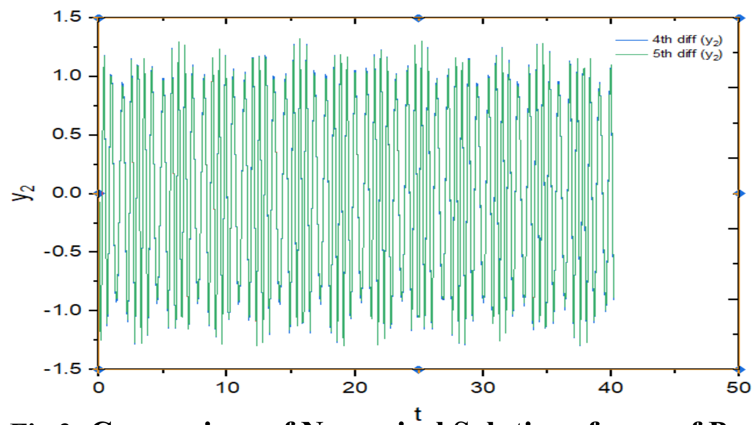

Fig 3: Comparison of Numerical Solutions for y_2 of Problem 1

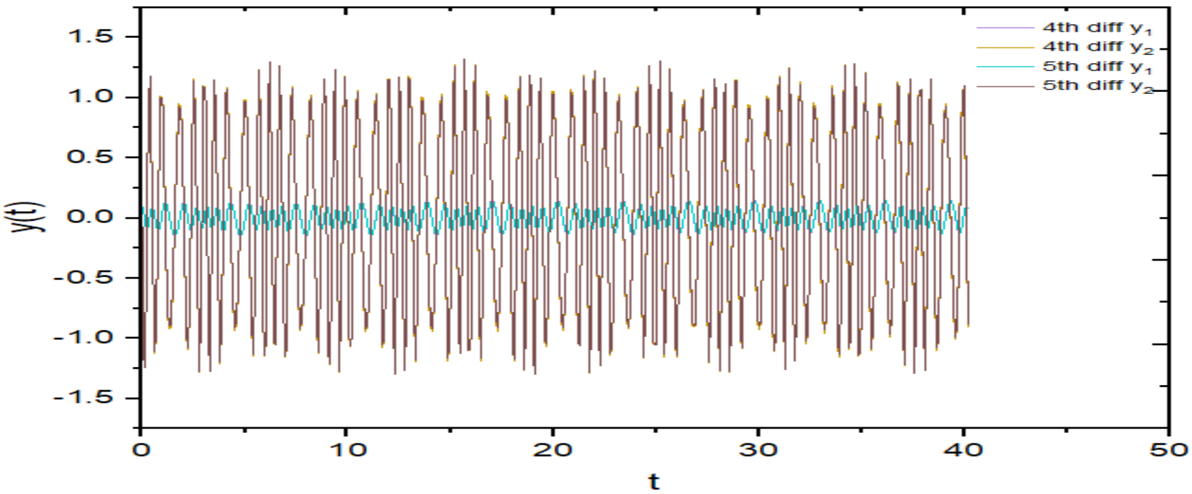

Fig 4: Comparison of Numerical Solutions for y_1 and y_2 of Problem 1

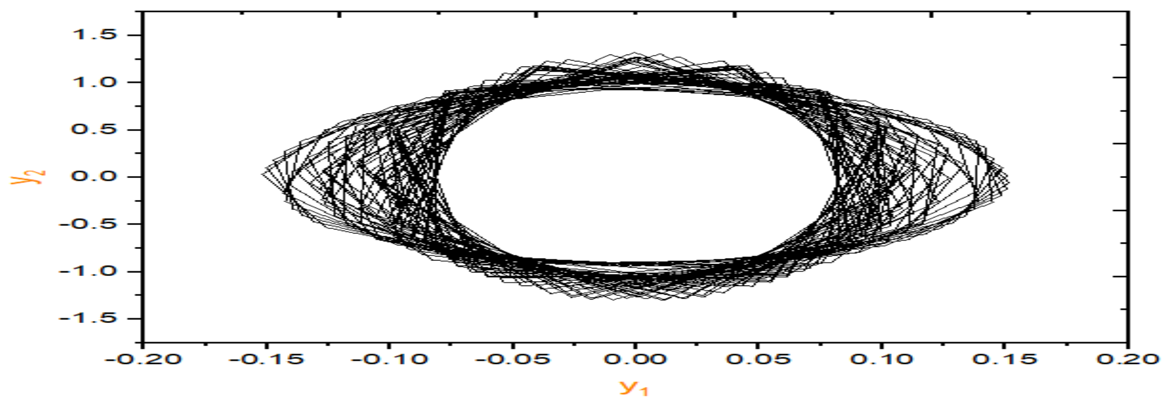

Fig 4.6: Phase portrait for Problem 1

3.3 Examples from Population Dynamics

Problem 2: Consider the Lotka-Volterra equations.
The Lotka-Volterra equations are a classic mathematical model used to describe the dynamics of predator-prey interactions in ecology.

$$y_1' = y_1 - y_1 y_2, \quad (3.4)$$

$$y_2' = \frac{-1}{5} y_2 + y_1 y_2 \quad (3.5)$$

Where the initial conditions $y_1(0)=1$, $y_2(0)=1$.

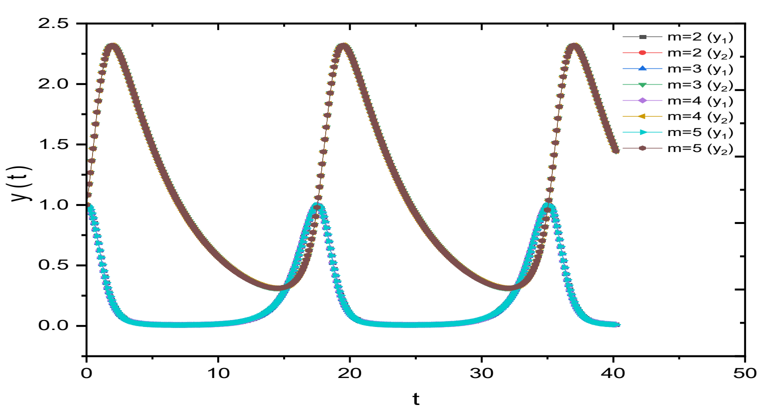

Fig 4.9: Comparison of Numerical Solutions foy y_1 of example 2

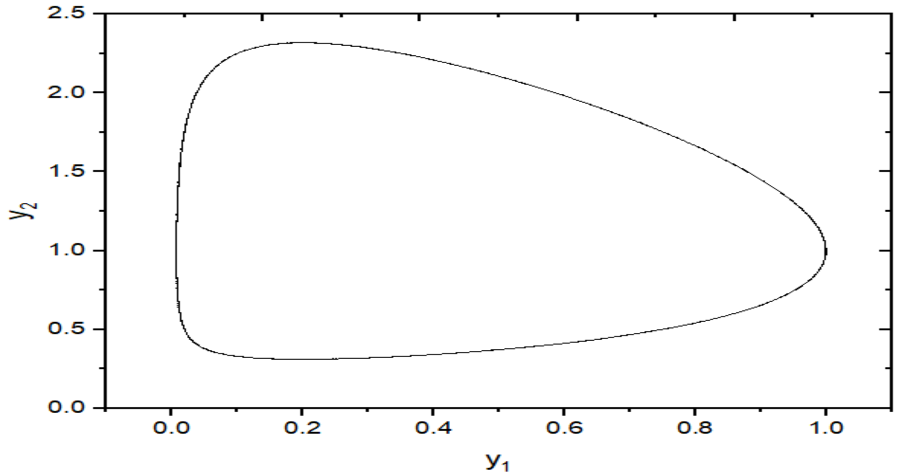

Fig 4.10: y_2 against y_1 for example 2

The plot shows the phase portrait of the Lotka-Volterra model, which describes the dynamics of predator-prey interactions. The phase portrait is a graphical representation of the system's behavior, where the horizontal axis represents the prey population y_1 and the vertical axis represents the predator population y_2. The closed curve in the phase portrait indicates a stable limit cycle, which means that the system exhibits periodic oscillations.

3.4 Examples from Chaos Theory
Problem 3

The Lorenz system is known for its chaotic solutions, displaying sensitivity to initial conditions. Its behavior is often visualized in phase space diagrams or attractor plots, revealing intricate and seemingly unpredictable patterns.

$$y_1' = a(y_2 - y_1). \quad (3.6)$$
$$y_2' = -y_1 y_2 - b y_1 - y_2 \quad (3.7)$$
$$y_3' = y_1 y_2 - c y_3 \quad (3.8)$$

Subject to $y_1(0)=1, y_2(0)=5, y_3(0)=10$ and the constants are $a=10, b=28$ and $c=\frac{8}{3}$

These values are chosen specifically for the Lorenz system to exhibit chaotic behavior, wherein small differences in initial conditions lead to significantly different trajectories over time.

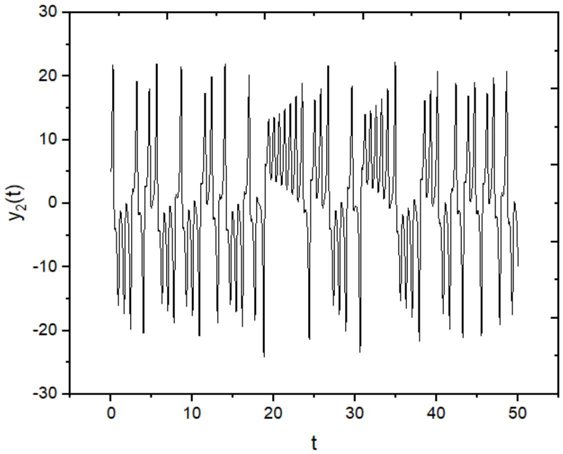

Fig 7: Numerical solution of y_1 against t of problem 3

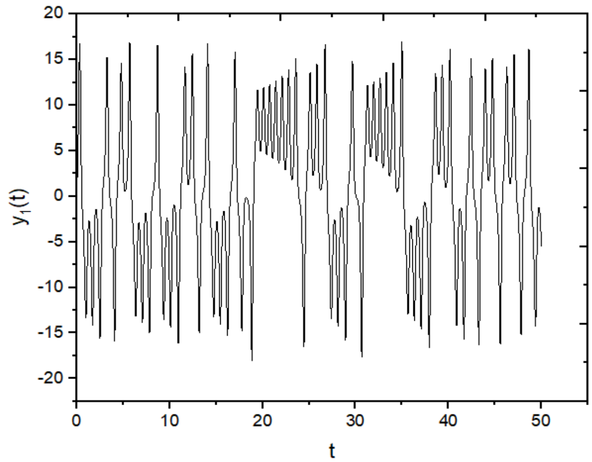

Fig 8: Numerical solution of y_2 against t of problem 3

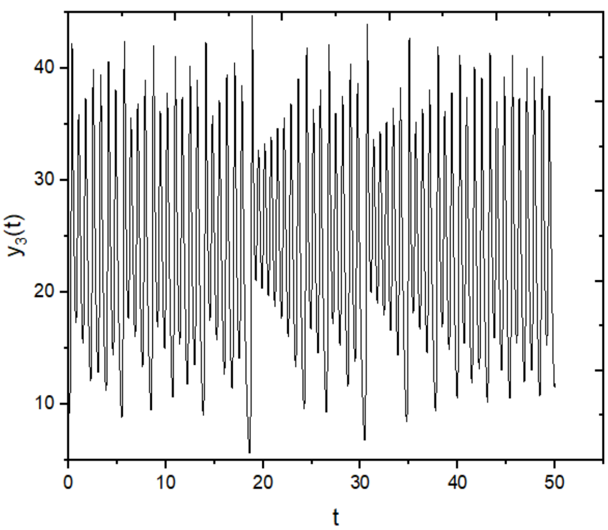

Fig 9: Numerical solution of y_3 against t problem 3

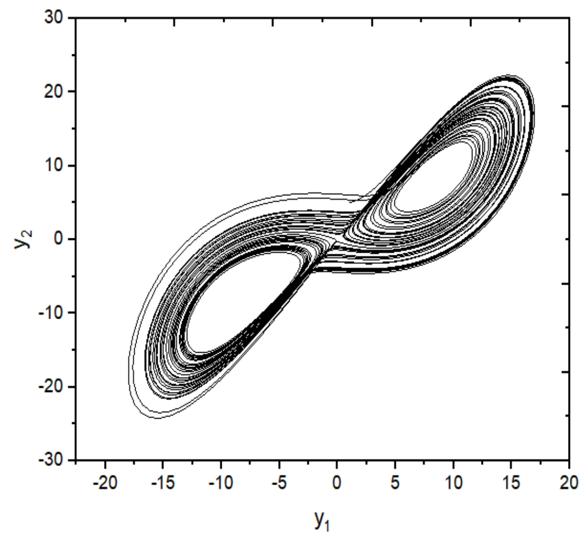

Fig 10: Phase portrait of y_2 against y_1 problem 3

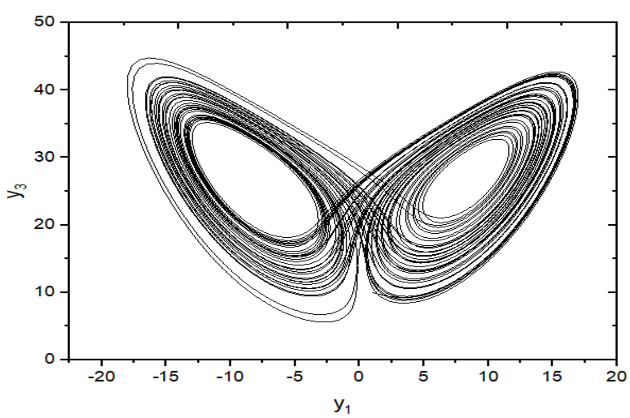

Fig 11: Phase portrait of y_3 against y_1 of problem 3

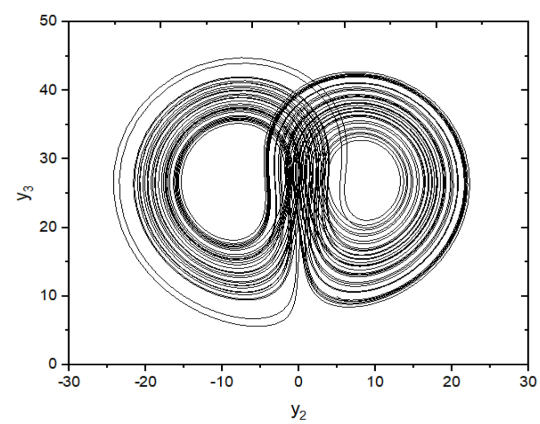

Fig 12: Phase portrait of y_3 against y_2 of problem 3

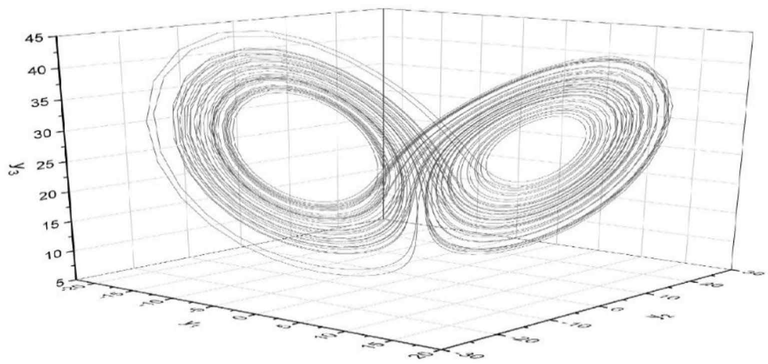

Fig 13: Phase portrait of Problem 3

Problem 4

Next, we consider the Rossler system governed by

$$y_1' = -y_2 - y_3, \quad (3.9)$$
$$y_2' = y_1 + a y_2, \quad (3.10)$$
$$y_3' = b + y_3(y_1 - c), \quad (3.11)$$

A chaotic attractor with $a=0.2$, $b=0.2$, and $c=5.7$. The Rossler system's attractor resulting from these equations with the given parameters forms a distinctive shape often referred to as the Rossler attractor.

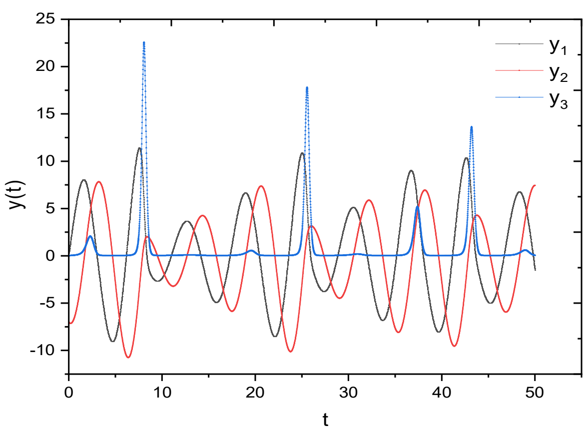

Fig 14: Numerical solutions of $y(t)$ against t of problem 4

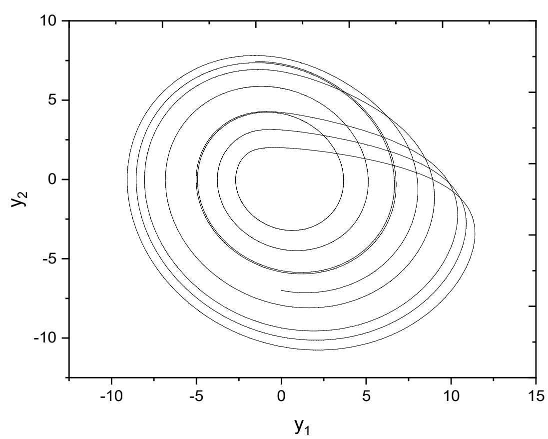

Fig 15: Phase portrait of y_2 against y_1 of problem 4

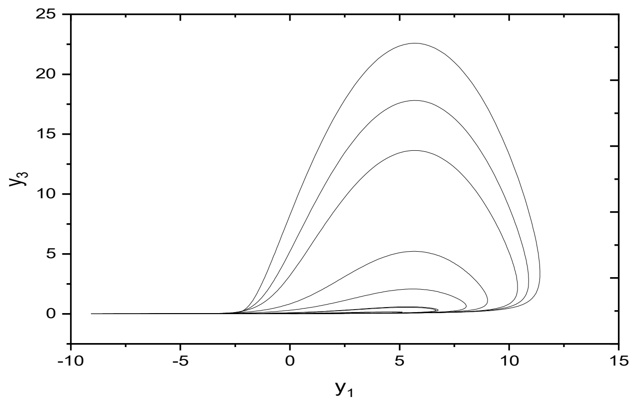

Fig 16: Phase portrait of y_3 against y_1 of problem 3

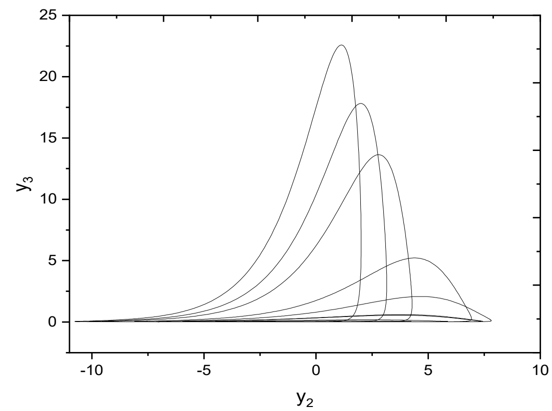

Fig 17: Phase portrait of y_3 against y_2 of problem 3

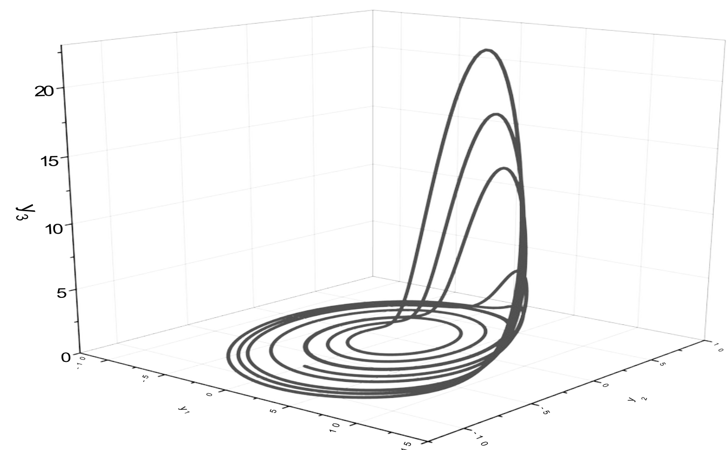

Fig 18: Phase portrait of Problem 4

Conclusion:

In conclusion, this paper presents two innovative two-step implicit hybrid block multistep methods incorporating multi-derivatives for the numerical solution of nonlinear dynamical first-order ordinary differential equations. By leveraging the Bhaskara cosine approximation formula to generate hybrid points and using power series as the basis function, these methods achieve higher order, zero-stability, consistency, and A-stability. The effectiveness and reliability of these methods are demonstrated through their application to problems from chaos theory, the SIR model, and multi-dimensional problems. The results indicate that these methods provide superior solutions compared to existing techniques, highlighting their potential for broader use in complex nonlinear systems.

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