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HYERS-ULAM STABILITY OF NONAUTONOMOUS THIRD ORDER NONLINEAR DIFFERENTIAL EQUATIONS.

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ABSTRACT

Hyers-Ulam stability of non-autonomous third order nonlinear differential equations is considered in this paper. This consideration is possible by using the Bihari integral inequality and Gronwall-Bellman-Bihari integral inequality to prove Hyers-Ulam stability and determine Hyers-Ulam constant of every non-autonomous third order nonlinear differential equation considered. Our results improve and extend known results in literature.

1. Introduction

In this paper, we consider the Hyers-Ulam stability of the following non-autonomous nonlinear third order ordinary differential equations given as:

$$u'''(t) + \beta(t)f(u(t))u''(t) + \alpha(t)g(u(t))u'(t) + \rho(t)\gamma(u(t)) = P(t,u(t),u'(t)), \quad (1)$$

$$u'''(t) + \beta(t)f(u(t))u''(t) + \rho(t)\gamma(u(t)) = P(t,u(t),u'(t)), \quad (2)$$

$$u'''(t) + \alpha(t)g(u(t))u'(t) + \rho(t)\gamma(u(t)) = P(t,u(t),u'(t)), \quad (3)$$

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$$u'''(t)+\beta(t)f(u(t))u''(t)+\alpha(t)g(u(t))u'(t)+\rho(t)\gamma(u(t))=0, \quad (4)$$

$$u'''(t)+\beta(t)f(u(t))u''(t)+\rho(t)\gamma(u(t))=0, \quad (5)$$

$$u'''(t)+\alpha(t)g(u(t))u'(t)+\rho(t)\gamma(u(t))=0, \quad (6)$$

for $t>0$, with initial value

$$u(t_0)=u'(t_0)=u''(t_0)=0, \quad (7)$$

where $u,\beta,\alpha,\rho\in C(\mathbf{I})$, $g,\gamma,f\in C(\mathbf{R}_+)$, $P\in C(\mathbf{I}\times\mathbf{R}^2)$, $\mathbf{I}=(t_0,\infty)$, $\mathbf{R}_+=[0,\infty)$ *and* $\mathbf{R}=(-\infty,\ \infty)$, $P(t_0,0,0)=0$.

The Hyers-Ulam stability of non-autonomous third order differential equations (1),(2),(3),(4),(5) and (6) have not been considered in the literature. Some researchers only studied the Hyers-Ulam stability of first, second and third order linear differential equations see[2, 10, 12, 13, 14, 15, 16, 17, 19, 20,21, 27,28]. While the following researchers investigated Hyers-Ulam stability of first and second order nonlinear differential equations see[1, 7, 8, 9, 22, 23, 24, 25, 26].

Motivation for this work comes from the papers in [5,6,7], where Hyers-Ulam and Hyers-Ulam-Rassias stability of second order nonlinear differential equations were proved. This article extends these papers to third order nonlinear differential equations.

2 DEFINITIONS

The following definitions are given for the purpose of establishing our results:

2.1 Definition: A function $\omega:[0,\infty)\to[0,\infty)$ is said to belong to a class Ψ if

1. $\omega(u)$ is nondecreasing and continuous for $u\geq 0$,
2. $(\frac{1}{v})\omega(u)\leq\omega(\frac{u}{v})$ for all u *and* $v\geq 1$,
3. there exists a function ϕ, continuous on $[0,\infty)$ with $\omega(\alpha u)\leq\phi(\alpha)\omega(u)$ for $\alpha\geq 0$.

2.2 Definition: We say equation (1) has the Hyers-Ulam stability, if there exists a constant $K_1\geq 0$ with following property: for every $\epsilon>0$, $u(t)\in C^3(\mathbf{R}_+)$, if

$$|u'''(t)+\beta(t)f(u(t))u''(t)+\alpha(t)g(u(t))u'(t)+\rho(t)\gamma(u(t))-P(t,u(t),u'(t))|\leq\epsilon, \quad (8)$$

then there exists some $u_0\in C^3(\mathbf{R}_+)$ satisfying equation (1) such that

$$|u(t)-u_0(t)|\leq K_1\epsilon,$$

where K_1 is called Hyers-Ulam constant for equation (1).

2.3 Definition: We say equation (2) has the Hyers-Ulam stability, if there exists a constant $K_2\geq 0$ with following property: for every $\epsilon>0$, $u(t)\in C^3(\mathbf{R}_+$,if

$$|u'''(t)+\beta(t)f(u(t))u''(t)+\rho(t)\gamma(u(t))-P(t,u(t),u'(t))|\leq\epsilon, \quad (9)$$

then there exists some $u_0\in C^3(\mathbf{R}_+)$ satisfying equation (2) such that

$$|u(t)-u_0(t)|\leq K_2\epsilon,$$

we call such K_2 Hyers-Ulam constant for equation (2).

2.4 Definition: The differential equation (3) has the Hyers-Ulam stability with initial condition (8), if there exists a positive constant $K_3\geq 0$ with following property: for every $\epsilon>0$, $u(t)\in C^3(\mathbf{R}_+)$, which satisfies

$$|u'''(t)+\alpha(t)g(u(t))u'(t)+\rho(t)\gamma(u(t))-P(t,u(t),u'(t))|\leq\epsilon, \quad (10)$$

then there exists a function $u_0 \in C^3(\mathbf{R}_+)$ satisfying equation (3) with initial conditions (8) such that

$$|u(t) - u_0(t)| \le K_3\epsilon,$$

where K_3 is called Hyers-Ulam constant for equation (3) with initial conditions (7).

2.5 Definition: The differential equation (4) is stable in the sense of Hyers-Ulam, if there exists $K_4 \ge 0, \quad \epsilon > 0$ and $u(t) \in C^3(\mathbf{R}_+)$ satisfying

$$|u'''(t) + \beta(t)f(u(t))u''(t) + \alpha(t)g(u(t))u'(t) + \rho(t)\gamma(u(t))| \le \epsilon, \tag{11}$$

whenever the solution $u_0(t) \in C^3(\mathbf{R}_+)$ of the equation (4) satisfies

$$|u(t) - u_0(t)| \le K_4\epsilon,$$

where K_4 is called a Hyers-Ulam constant for equation (4) with initial conditions (7).

2.6 Definitions: We say equation (5) with initial conditions (7) has the Hyers-Ulam stability, if there exists a constant $K_5 > 0$ with following property: for every $\epsilon > 0$, $u(t) \in C^3(\mathbf{R}_+)$,if

$$|u'''(t) + \beta(t)f(u(t))u''(t) + \rho(t)\gamma(u(t))| \le \epsilon, \tag{12}$$

then there exists some $u \in C^3(\mathbf{R}_+)$ satisfying equation (5) such that

$$|u(t) - u_0(t)| \le K_5\epsilon,$$

where K_5 is called Hyers-Ulam constant for equation (5) with initial conditions.

2.6 Definition: The differential equation (6) is stable in the sense of Hyers-Ulam, if there exists $K_6 \ge 0, \quad \epsilon > 0$ and $u(t) \in C^3(\mathbf{I}, \mathbf{R}_+)$ satisfying

$$|u'''(t) + \alpha(t)g(u(t))u'(t) + \rho(t)\gamma(u(t))| \le \epsilon, \tag{13}$$

whenever the solution $u_0(t) \in C^3(\mathbf{R}_+)$ of the equation (6) satisfies

$$|u(t) - u_0(t)| \le K_6\epsilon,$$

where K_6 is called a Hyers-Ulam constant for equation (6) with initial conditions (7).

3 LEMMAS AND THEOREMS

The following lemmas and theorems are needful to establish our claims.

3.1 Lemma [3]: Let $u(t)$, $f(t)$ be positive continuous functions defined on $a \le t \le b, (\le \infty)$ and $K > 0$, $M \ge 0$, further let $\omega(u)$ be a nonnegative nondecreasing continuous function for $u \ge 0$, then the inequality

$$u(t) \le K + M\int_{t_0}^{t} f(s)\omega(u(s))ds, \quad t_0 \le t < b, \tag{14}$$

implies the inequality

$$u(t) \le \Omega^{-1}\left(\Omega(k) + M\int_{t_0}^{t} f(s)ds\right), \quad t_0 \le t \le b' \le b. \tag{15}$$

Where

$$\Omega(u) = \int_{u_0}^{u} \frac{dt}{\omega(t)}, \qquad 0 < u_0 < u. \tag{16}$$

In the case $\omega(0) > 0$ or $\Omega(0+)$ is finite, one may take $u_0 = 0$ and Ω^{-1} is the inverse function of Ω and t must be in the subinterval $[t_0, b']$ of $[t_0, b]$ such that

$$\Omega(k) + M\int_{t_0}^{t} f(s)ds \in Dom(\Omega^{-1}).$$

3.2 Lemma [11]: Let $r(t)$ be an integrable function then the n-successive integration of r over the interval $[t_0, t]$ is given by

$$\int_{t_0}^{t} \ldots \int_{t_0}^{t} r(s)ds^n = \frac{1}{(n-1)!}\int_{t_0}^{t} (t-s)^{n-1}r(s)ds. \tag{17}$$

3.1 Theorem[18]: If $f(t)$ *and* $g(t)$ are continuous in $[t_0, t] \subseteq \mathbf{I}$ *and* $f(t)$ does not change sign in the interval, then there is a point $\xi \in [t_0, t]$ such that $\int_{t_0}^{t} g(s)f(s)ds = g(\xi)\int_{t_0}^{t} f(s)ds$.

3.2 Theorem [4] : Let

1. $u(t), r(t)\colon (0,\infty) \to (0,\infty)$ and continuous on $(0,\infty)$,
2. $\varpi \in \Psi$,
3. $n > 0$ be monotonic, nondecreasing and continuous on $(0,\infty)$,

if

$$u(t) \leq n(t) + \int_0^t f(s)\varpi(u(s))ds, \quad 0 < t < \infty, \tag{18}$$

then

$$u(t) \leq n(t)\Omega^{-1}\left(\Omega(1) + \int_0^t f(s)ds\right), \; 0 < t \leq b, \tag{19}$$

where $(0, b) \subset (0, \infty)$, where $\Omega(u)$ is defined in (9) and Ω^{-1} is the inverse of Ω and t is in the subinterval $(0, b)$ is so chosen that

$$\Omega(1) + \int_0^t f(s)ds \in Dom(\Omega^{-1}).$$

3.3 Theorem [5, 6]: Suppose $u(t), r(t), h(t) \in C(\mathbf{I}, \mathbf{R}_+)$ and $\varpi(u), \beta(u) \in \Psi$ are nonnegative, monotonic, nondecreasing, continuous and $\omega(u)$ be submultiplicative for $u > 0$. Let

$$u(t) \leq K + T\int_{t_0}^{t} r(s)\beta(u(s))ds + L\int_{t_0}^{t} h(s)\varpi(u(s))ds, \tag{20}$$

for $K,\ T$ *and* L are positive constants,

$$\begin{aligned} u(t) \leq \Omega^{-1}&\left(\Omega(K) + L\int_{t_0}^{t} h(s)\varpi\left(F^{-1}\left(F(1) + T\int_{t_0}^{s} r(\alpha)d\alpha\right)\right)ds\right) \\ &F^{-1}\left(F(1) + T\int_{t_0}^{t} r(s)ds\right), \end{aligned} \tag{21}$$

where $\beta(u) \neq \varpi(u)$, Ω is defined in equation (16) and $F(u)$ is defined as

$$F(u) = \int_{u_0}^{u} \frac{ds}{\beta(s)}, \quad 0 < u_0 \leq u, \tag{22}$$

F^{-1}, Ω^{-1} are the inverses of F, Ω respectively and t is in the subinterval $(0, b) \in \mathbf{I}$ so that

$$F(1) + T\int_{t_0}^{t} r(s)ds \in Dom(F^{-1})$$

and

$$\Omega(K) + L\int_{t_0}^{t} h(s)\varpi\left(F^{-1}\left(F(1) + T\int_{t_0}^{t} r(\alpha)d\alpha\right)\right)ds, \in Dom(\Omega^{-1}).$$

3.4 Theorem [5, 6]: If $u(t), r(t), h(t), \rho(t), g(t) \in C(\mathbf{R}_+)$ and $\omega, f, \gamma \in \Psi$ be nonnegative, monotonic, nondecreasing continuous functions. Let γ be submultiplicative. If

$$\begin{aligned} u(t) \leq \rho(t) + A\int_{t_0}^{t} r(s)\beta(u(s))ds + B\int_{t_0}^{t} h(s)\varpi(u(s))ds + \\ L\int_{t_0}^{t} g(s)\gamma(u(s))ds \end{aligned} \tag{23}$$

for $K, A, B, L > 0$, then

$$\begin{aligned} &u(t) \leq \rho(t)\Upsilon^{-1} \\ &\left[\Upsilon(1) + L\int_{t_0}^{t} g(s)\gamma\left[\Omega^{-1}\left(\Omega(1) + B\int_{t_0}^{s} h(\alpha)\varpi(T(\alpha))d\alpha\right)T(s)\right]ds\right] \\ &\Omega^{-1}\left(\Omega(1) + B\int_{t_0}^{t} h(s)\varpi(T(s))ds\right)T(t) \end{aligned} \tag{24}$$

where $T(t)$ is given as

$$T(t) = F^{-1}\left(F(1) + A\int_{t_0}^{t} r(s)ds\right) \qquad (25)$$

and

$$\Upsilon(r) = \int_{t_0}^{t} \frac{ds}{\gamma(s)}, \quad 0 < r_0 \le r, \qquad (26)$$

and F^{-1}, Ω^{-1} and Υ^{-1} are the inverses of F, Ω, Υ respectively $t \in (0,b) \subset (I)$. So that

$$\Upsilon(1) + L\int_{t_0}^{t} g(s)\gamma\left[\Omega^{-1}\left(\Omega(1) + B\int_{t_0}^{s} h(\alpha)\varpi(T(\alpha))d\alpha\right)T(s)\right]ds \in Dom(\Upsilon^{-1}).$$

MAIN RESULTS

We begin this section with the investigation of Hyers-Ulam stability of equation (1).

4.1 Theorem : Suppose:

i. $|u''(t)| \le \delta$, where $\delta > 0$,

ii. $|u'''(\xi)| \le \psi$, where $\psi > 0$,

iii. $|u'(t)| \le \lambda$ where $\lambda > 0$,

iv. $\int_{t_0}^{t} |u'(s)|ds \le \varrho$, where $\varrho > 0$,

v. $|P(t,u(t),u'(t))| \le \varphi(t)\phi(u(t))h(|u'(t)|)$ where $\varphi, h, \phi \in C(\mathbf{R}_+)$ and h, ϕ belong to class Ψ,

vi. $U(u(t)) = \int_{u(t_0)}^{u(t)} f(s)ds$,

vii. let $\beta(t)$ a nondecreasing function, then, $\beta' \ge 0$ implies there exits $\sigma > 0$ such that $\beta(t) \ge \sigma$,

viii. $|U(u(t))| \ge |u(t)|$,

ix. $\lim_{t\to\infty}\int_{t_0}^{t} \varphi(s)ds \le d$ where $d > 0$,

x. $\lim_{t\to\infty}\int_{t_0}^{t} \alpha(s)ds \le m$, where $m > 0$,

xi. $\lim_{t\to\infty}\int_{t_0}^{t} \rho(s)ds \le n$, where $n > 0$,

are satisfied, then, nonlinear differential equation (1) is Hyers-Ulam stable with Hyers-Ulam constant given by

$$K_1 = \frac{\varrho(1+\psi)}{\sigma\delta}\left[\Upsilon^{-1}\left(\Upsilon(1) + d\frac{h(\lambda)\lambda}{\sigma\delta}\phi(\Omega^{-1}(\Omega(1) + m\frac{\lambda}{\sigma\delta}\gamma(T^*)\right)T^*\right)\right)\Big] \qquad (27)$$
$$\Omega^{-1}\left(\Omega(1) + m\frac{\lambda}{\sigma\delta}\gamma(T^*)\right)T^*.$$

Proof: From inequality (8), we obtain

$$-\epsilon \le u'''(t) + \beta(t)f(u(t))u''(t) + \alpha(t)g(u(t))u'(t) + \rho(t)\gamma(u(t)) - P(t,u(t),u'(t) \le \epsilon. \qquad (28)$$

Multiplying (28) by $u'(t)$, we obtain

$$-u'(t)\epsilon \le u'''(t)u'(t) + \beta(t)f(u(t))u''(t)u'(t) + \alpha(t)g(u(t))(u'(t))^2 + \rho(t)\gamma(u(t))u'(t) - P(t,u(t),u'(t))u'(t) \le u'(t)\epsilon. \qquad (29)$$

Integrating (29) from t_0 to t thrice and applying Lemma 3.1, we have

$$-t^2\epsilon\int_{t_0}^{t} u'(s)ds \leq t^2\int_{t_0}^{t} u'''(s)u'(s)ds + t^2\int_{t_0}^{t} \beta(s)f(u(s))u''(s)u'(s)ds$$
$$+t^2\int_{t_0}^{t} \alpha(s)g(u(s))(u'(s))^2ds + t^2\int_{t_0}^{t} \rho(s)\gamma(u(s))u'(s)ds \quad (30)$$
$$-t^2\int_{t_0}^{t} P(s,u(s),u'(s))u'(s)ds \leq t^2\epsilon\int_{t_0}^{t} u'(s)ds,\ \forall t > 0.$$

Consider inequality (30) in the form

$$t^2\int_{t_0}^{t} u'''(s)u'(s)ds + t^2\int_{t_0}^{t} \beta(s)f(u(s))u''(s)u'(s)ds + t^2\int_{t_0}^{t} \alpha(s)g(u(s))(u'(s))^2ds$$
$$+t^2\int_{t_0}^{t} \rho(s)\gamma(u(s))u'(s)ds - t^2\int_{t_0}^{t} P(s,u(s),u'(s))u'(s)ds \leq t^2\epsilon\int_{t_0}^{t} u'(s)ds. \quad (31)$$

For $t > 0$, multiplying by $\frac{2}{t^2}$ and applying Theorem 3.1 implies there exists $\xi \in [t_0, t]$ such that

$$u'''(\xi)\int_{t_0}^{t} u'(s)ds + t^2\int_{t_0}^{t} \beta(s)f(u(s))u''(s)u'(s)ds + t^2\int_{t_0}^{t} \alpha(s)g(u(s))(u'(s))^2ds$$
$$+t^2\int_{t_0}^{t} \rho(s)\gamma(u(s))u'(s)ds - t^2\int_{t_0}^{t} P(s,u(s),u'(s))u'(s)ds \leq t^2\epsilon\int_{t_0}^{t} u'(s)ds.$$
(32)

We use condition (vi) to get

$$u'''(\xi)\int_{t_0}^{t} u'(s)ds + \int_{t_0}^{t} \beta(s)u''(s)\frac{d}{ds}U(u(s))ds + \int_{t_0}^{t} \alpha(s)g(u(s))(u'(s))^2ds$$
$$+\int_{t_0}^{t} \rho(s)\gamma(u(s))u'(s)ds - \int_{t_0}^{t} P(s,u(s),u'(t))u'(s)ds \leq \epsilon\int_{t_0}^{t} u'(s)ds, \quad (33)$$

and apply condition (vii) of Theorem 4.1 to obtain

$$\sigma|u''(t)||U(u(t))| \leq \epsilon\int_{t_0}^{t} |u'(s)|ds + |u'''(\xi)|\int_{t_0}^{t} u'(s)ds + |u'(t)|^2\int_{t_0}^{t} \alpha(s)g(|u(s)|)ds$$
$$+|u'(t)|\int_{t_0}^{t} \rho(s)\gamma(u(s))ds + |u'(t)|\int_{t_0}^{t} |P(s,u(s),u'(s))|ds. \quad (34)$$

Using conditions (i),(ii),(iii),(iv),(v) and (viii) of Theorem 4.1 to arrive at

$$|u(t)| \leq N + \frac{\lambda^2}{\sigma\delta}\int_{t_0}^{t} \alpha(s)g(|u(s)|)ds + \frac{\lambda}{\sigma\delta}\int_{t_0}^{t} \rho(s)\gamma(|u(s)|)ds +$$
$$\frac{\hbar(\lambda)\lambda}{\sigma\delta}\int_{t_0}^{t} \varphi(s)\phi(|u(s)|)ds, \quad (35)$$

where

$$N = \epsilon\frac{\varrho(1+\psi)}{\sigma\delta}. \quad (36)$$

Applying Theorem 3.4.to inequality (36) by leting $p(t) = N$, we get

$$|u(t)| \leq N\Big[\Upsilon^{-1}\Big(\Upsilon(1) + \frac{\hbar(\lambda)\lambda}{\sigma\delta}\int_{t_0}^{t} \varphi(s)\phi(\Omega^{-1}(\Omega(1)$$
$$+\frac{\lambda}{\sigma\delta}\int_{t_0}^{s} \rho(\alpha)\gamma(T(\alpha))d\alpha\Big)T(s)\Big)ds\Big)\Big] \quad (37)$$
$$\Omega^{-1}\Big(\Omega(1) + \frac{\lambda}{\sigma\delta}\int_{t_0}^{t} \rho(s)\gamma(T(s))ds\Big)T(t),$$

where

$$T(t) = F^{-1}\Big(F(1) + \frac{\lambda^2}{\sigma\delta}\int_{t_0}^{t} \alpha(s)ds\Big) \quad (38)$$

Using conditions (ix),(x) and (xi) to obtain

$$|u(t)| \leq N\Big[\Upsilon^{-1}\Big(\Upsilon(1) + d\frac{h(\lambda)\lambda}{\sigma\delta}\phi(\Omega^{-1}(\Omega(1) + m\frac{\lambda}{\sigma\delta}\gamma(T^*)\Big)T^*\Big)\Big)\Big] \Omega^{-1}\Big(\Omega(1) + m\frac{\lambda}{\sigma\delta}\gamma(T^*)\Big)T^*, \tag{39}$$

where

$$T^* = F^{-1}\Big(F(1) + n\frac{\lambda^2}{\sigma\delta}\Big). \tag{40}$$

Substituting for N using equation (36) in inequality (39) we have

$$|u(t)| \leq \epsilon\frac{\varrho(1+\psi)}{\sigma\delta}\Big[\Upsilon^{-1}\Big(\Upsilon(1) + d\frac{h(\lambda)\lambda}{\sigma\delta}\phi(\Omega^{-1}(\Omega(1) + m\frac{\lambda}{\sigma\delta}\gamma(T^*)\Big)T^*\Big)\Big)\Big] \Omega^{-1}\Big(\Omega(1) + m\frac{\lambda}{\sigma\delta}\gamma(T^*)\Big)T^*. \tag{41}$$

Hence,

$$|u(t) - u_0(t)| \leq |u(t)| \leq \epsilon\frac{\varrho(1+\psi)}{\sigma\delta}\Big[\Upsilon^{-1}\Big(\Upsilon(1) + d\frac{h(\lambda)\lambda}{\sigma\delta}\phi(\Omega^{-1}(\Omega(1) + m\frac{\lambda}{\sigma\delta}\gamma(T^*)\Big)T^*\Big)\Big)\Big] \Omega^{-1}\Big(\Omega(1) + m\frac{\lambda}{\sigma\delta}\gamma(T^*)\Big)T^*. \tag{42}$$

Therefore,

$$|u(t) - u_0(t)| \leq \epsilon K_1,$$

where K_1 (Hyers-Ulam constant) is given as

$$K_1 = \frac{\varrho(1+\psi)}{\sigma\delta}\Big[\Upsilon^{-1}\Big(\Upsilon(1) + d\frac{h(\lambda)\lambda}{\sigma\delta}\phi\Big(\Omega^{-1}\Big(\Omega(1) + m\frac{\lambda}{\sigma\delta}\gamma(T^*)\Big)T^*\Big)\Big)\Big] \Omega^{-1}\Big(\Omega(1) + m\frac{\lambda}{\sigma\delta}\gamma(T^*)\Big)T^*.$$

4.1 Example: Consider the Hyers-Ulm stability of the third order nonlinear differential equation of the form

$$u'''(t) + t^2u^2(t)u''(t) + t^{-2}u(t)u'(t) + t^{-3}u^4(t) = t^{-4}u^2(t) \quad t > 0$$

where

$$P(t, u(t)) = t^{-4}u^2(t) \leq t^{-2}u^2(t)$$

with the initial condition(7). By conditions of Theorem 4.1, non-autonomous third order nonlinear differential equation is Hyers-Ulam stable.

4.2 Theorem: Let all the conditions of Theorem 4.1 remain valid. Then a nonlinear differential equation (2) has Hyers-Ulam stability property with Hyers-Ulam constant given by

$$K_2 = \frac{\varrho(1+\psi)}{\delta\sigma}\Omega^{-1}\left(\Omega(1) + d\lambda h(\lambda)\phi\left(F^{-1}\left(F(1) + \frac{n\lambda}{\delta\sigma}\right)\right)\right) \left(F(1) + \frac{n\lambda}{\delta\sigma}\right). \tag{43}$$

Proof. Using inequality (9), multiplying both sides by $u'(t)$ and we consider

$$u'''(t)u'(t) + \beta(t)f(u(t))u''(t)u'(t) + \rho(t)\gamma(u(t))u'(t) - P(t, u(t), u'(t))u'(t) \leq \epsilon u'(t) \tag{44}$$

Integrating from t_0 to t thrice, using Lemma 3.2 and condition (vi) of Theorem 4.1 to get

$$t^2 \int_{t_0}^{t} u'''(s)u'(s)ds + \int_{t_0}^{t} \beta(s)u''(s)\frac{d}{ds}U(u(s))ds$$
$$+t^2\rho(t)\gamma(u(t))u'(t) - t^2 \int_{t_0}^{t} P(s,u(s),u'(s))u'(s)ds \leq t^2\epsilon \int_{t_0}^{t} u'(s)ds. \quad (45)$$

Empolying Theorem 3.2, there exists $\xi \in [t_0, t]$ such that

$$u'''(\xi)\int_{t_0}^{t} u'(s)ds + \int_{t_0}^{t} \beta(s)u''(s)\frac{d}{ds}U(u(s))ds$$
$$+\int_{t_0}^{t} \rho(s)\gamma(u(s))u'(s)ds - \int_{t_0}^{t} P(s,u(s),u'(t))u'(s)ds \leq \epsilon \int_{t_0}^{t} u'(s)ds, \ \forall t > 0. \quad (46)$$

Using condition (viii) of Theorem 4.1 and integration by parts to get

$$u''(t)\sigma U(u(t)) \leq \epsilon \int_{t_0}^{t} u'(s)ds - u'''(\xi)\int_{t_0}^{t} u'(s)ds$$
$$-\int_{t_0}^{t} u'(s)\rho(s)\gamma(u(s))ds + \int_{t_0}^{t} P(s,u(s),u'(s))u'(s)ds. \quad (47)$$

Applying the conditions (i), (ii),(iii),(iv),(v) and (vii) of Theorem 4.1, we obtain

$$|u(t)| \leq \epsilon\varrho\frac{(1+\psi)}{\delta\sigma} + \frac{\lambda}{\delta\sigma}\int_{t_0}^{t} \rho(s)\gamma(|u(s)|)ds + \frac{\lambda h(\lambda)}{\delta\sigma}\int_{t_0}^{t} \varphi(s)\phi(|u(s)|)ds. \quad (48)$$

By applying Theorem 3.3 we get

$$|u(t)| \leq \epsilon\frac{\varrho(1+\psi)}{\delta\sigma}\Omega^{-1}\Big(\Omega(1) + \frac{\lambda h(\lambda)}{\delta\sigma}\int_{t_0}^{t} \varphi(s)\phi(F^{-1}(F(1)$$
$$+\frac{\lambda}{\delta\sigma}\int_{t_0}^{t} \rho(\alpha)d\alpha\Big)ds\Big)\Big) \quad (49)$$
$$F^{-1}\Big(F(1) + \frac{\lambda}{\delta\sigma}\int_{t_0}^{t} \rho(s)ds\Big),$$

and the conditions (ix) and (xi) to have

$$|u(t)| \leq \epsilon\frac{\varrho(1+\psi)}{\delta\sigma}\Omega^{-1}\Big(\Omega(1) + \frac{\lambda h(\lambda)d}{\delta\sigma}\phi(F^{-1}(F(1)$$
$$+\frac{n\lambda}{\delta\sigma}\Big)\Big)\Big)F^{-1}\Big(F(1) + \frac{n\lambda}{\delta\sigma}\Big). \quad (50)$$

Hence,

$$|u(t) - u(t_0)| \leq |u(t)| \leq \epsilon K_2,$$

where

$$K_2 = \frac{\varrho(1+\psi)}{\delta\sigma}\Omega^{-1}\Big(\Omega(1) + \frac{\lambda h(\lambda)d\phi}{\delta\sigma}\Big(F^{-1}\Big(F(1) + \frac{n\lambda}{\delta\sigma}\Big)\Big)\Big)$$
$$F^{-1}\Big(F(1) + \frac{n\lambda}{\delta\sigma}\Big). \quad (51)$$

4.2 Example Investigate Hyers-Ulam Stability of the following third order nonlinear differential equation.

$$u'''(t) + t^{-4}u^4(t)u''(t) + t^{-2}u^2(t)u'(t) + t^{-6}u^2(t) = t^{-4}u^4(t)u'^2(t) \ \ t > 0$$

where $\beta(t) = t^{-4}$, $f(u(t)) = u^4(t)$, $\alpha(t) = t^{-2}$, $g(u(t)) = u^2(t)$, $\rho(t) = t^{-6}$, $\gamma u(t) = u^2(t)$ $P(t,u(t),u'(t)) = t^{-4}u^2(t)u'^2(t)$ where $\varphi(t) = t^{-2}$, $\phi(u(t)) = u^4(t)$ By carefully following the conditions of Theorem 4.2, the nonlinear differential equation has Hyers-Ulam stability.

4.3Theorem. The non-autonomous third order nonlinear differential equation (3) has Hyers-Ulam stability, if the conditions of Theorem 4.1 remain valid. In addition, let

i'. $R(u(t)) = \int_{u(t_0)}^{u(t)} r(u(s))ds$, ii'. $|R(u(t))| \geq |u(t)|$,

iii'. let ρ be an increasing function, then $\rho'(t) \geq 0$, implies that there exists $\xi > 0$ such that $\rho \geq \xi$,

hold, then equation (3) has Hyers-Ulam stability with Hyers-Ulam constant given as

$$K_3 = \varrho \frac{(1+\psi)}{\xi} \Omega^{-1} \left(\Omega(1) + d \frac{\lambda h(\lambda)}{\xi} \omega(F^{-1}(F(1) + n \frac{\lambda^2}{\xi})) \right) F^{-1} \left(F(1) + \frac{n\lambda^2}{\xi} \right). \tag{52}$$

Proof. We consider inequality (10) in the form

$$u'''(t) + \alpha(t) g(u(t)) u'(t) + \rho(t) r(u(t)) - P(t, u(t), u'(t)) \leq \epsilon, \tag{53}$$

and multiply inequality (53) by $u'(t)$, integrate from t_0 to t thrice and use Lemma 3.2 to obtain

$$t^2 \int_{t_0}^{t} u'''(s) u'(s) ds + t^2 \int_{t_0}^{t} \alpha(t) g(u(s)) (u'(s))^2 ds + t^2 \int_{t_0}^{t} \rho(s) \gamma(u(s)) u'(s) ds - t^2 \int_{t_0}^{t} P(s, u(s), u'(s)) u'(s) \leq \epsilon t^2 \int_{t_0}^{t} u'(s) ds. \tag{54}$$

Using conditions (I'), (iii') of Theorem 4.3 and integraton by part to obtain

$$\xi R(u(t)) \leq \epsilon \int_{t_0}^{t} u'(s) ds - u'''(t) \int_{t_0}^{t} u'(s) ds - (u'(t))^2 \int_{t_0}^{t} \alpha(s) g(u(s)) ds + u'(t) \int_{t_0}^{t} P(s, u(s), u'(s)) ds. \tag{55}$$

Taking the absolute value of both sides, using conditions (ii),(iii),(iv) of the Theorem 4.1 and condition (ii') of Theorem 4.3 to get

$$|u(t)|) \leq \frac{(1+\psi)}{\xi} \varrho \epsilon + \frac{(\lambda)^2}{\xi} \int_{t_0}^{t} \alpha(s) g(|u(s)|) ds + \frac{\lambda h(\lambda)}{\xi} \int_{t_0}^{t} \varphi(s) \omega(|u(s|)) ds. \tag{56}$$

Using Theorem 3.3 we get

$$|u(t)| \leq \varrho \epsilon \frac{(1+\psi)}{\xi} \Omega^{-1} \left(\Omega(1) + \frac{\lambda h(\lambda)}{\xi} \int_{t_0}^{t} \varphi(s) \omega(F^{-1}(F(1) + \frac{\lambda^2}{\xi} \int_{t_0}^{s} \alpha(\delta) d\delta)) ds \right) F^{-1} \left(F(1) + \frac{\lambda^2}{\xi} \int_{t_0}^{t} \alpha(s) ds \right), \quad t \in \mathbf{I}. \tag{57}$$

By conditions (ix) and (xi) of Theorem 4.1 we have

$$|u(t)| \leq \varrho \epsilon \frac{(1+\psi)}{\xi} \Omega^{-1} \left(\Omega(1) + d \frac{\lambda h(\lambda)}{\xi} \omega(F^{-1}(F(1) + n \frac{\lambda^2}{\xi})) \right) F^{-1} \left(F(1) + n \frac{\lambda^2}{\xi} \right). \tag{58}$$

Hence,

$$|u(t) - u_0(t)| \leq |u(t)| \leq K_3 \epsilon,$$

where

$$K_3 = \varrho \frac{(1+\psi)}{\xi} \Omega^{-1} \left(\Omega(1) + d \frac{\lambda h(\lambda)}{\xi} \omega(F^{-1}(F(1) + n \frac{\lambda^2}{\xi})) \right) F^{-1} \left(F(1) + n \frac{\lambda^2}{\xi} \right)$$

Example 4.3 Consider the Hyers-Ulm stability of the third order nonlinear differential equation of the form

$$u'''(t) + t^{-2}u^4(t)u''(t) + t^{-4}u(t) = t^{-4}u^2(t) \quad t > 0$$

where

$$P(t, u(t)) = t^{-4}u^2(t) \leq t^{-2}u^2(t)$$

with the initial conditions(7). By conditions of Theorem 2 that satisfied the Theorem 11(`)@, third order nonlinear differential equation is Hyers-Ulam stable.

4.4 Theorem: Let all conditions of Theorem 4.1 remain valid. Then, the non-autonomous third order nonlinear differential equation (4) has Hyers-Ulam stability with Hyers-Ulam constant given as

$$K_4 = \varrho \frac{(1+\psi)}{\delta\sigma} \Omega^{-1}\left(\Omega(1) + \frac{m\lambda}{\delta\sigma}\gamma\left(F^{-1}\left(F(1) + \frac{n\lambda^2}{\delta\sigma}\right)\right)\right) F^{-1}\left(F(1) + \frac{n\lambda^2}{\delta\sigma}\right). \tag{59}$$

Proof. It is clear from equation (11) after multiplying by $u'(t)$ that

$$\begin{aligned} &u'''(t)u'(t) + \beta(t)f(u(t))u''(t)u'(t) + \alpha(t)g(u(t))(u'(t))^2 + \rho(t)\gamma(u(t))u'(t) \\ &\leq u'(t)\epsilon \end{aligned} \tag{60}$$

and integrating from t_0 to t thrice and using Lemma 1.2 and condition (v) of Theorem 4.1 we get

$$\begin{aligned} &\int_{t_0}^{t} u'''(t)u'(s)ds + \int_{t_0}^{t} \beta(s)u''(s)\frac{d}{ds}U(u(s))ds + \int_{t_0}^{t} \alpha(s)g(u(s))(u'(s))^2 ds \\ &+ \int_{t_0}^{t} \rho(s)\gamma(u(s))u'(s)ds \leq \epsilon \int_{t_0}^{t} u'(s)ds. \end{aligned} \tag{61}$$

By Theorem 3.2 implies there exists $\xi \in [t_0, t]$ such that

$$\begin{aligned} &u'''(\xi)\int_{t_0}^{t} u'(s)ds + u''(t)\int_{t_0}^{t} \beta(s)\frac{d}{ds}U(u(s))ds + (u'(t))^2\int_{t_0}^{t} \alpha(s)g(u(s))ds \\ &+u'(s)\int_{t_0}^{t} \rho(s)\gamma(u(s))ds \leq \epsilon \int_{t_0}^{t} u'(s)ds. \end{aligned} \tag{62}$$

We use integration by parts and condition (iii') of Theorem 4.3 to obtain

$$\begin{aligned} &u'''(\xi)\int_{t_0}^{t} u'(s)ds + u''(t)\sigma U(u(s)) + (u'(t))^2\int_{t_0}^{t} \alpha(s)g(u(s))ds \\ &+ \int_{t_0}^{t} \rho(s)\gamma(u(s))u'(s)ds \leq \epsilon \int_{t_0}^{t} u'(s)ds. \end{aligned} \tag{63}$$

Applying the conditions (i)-(vi) of Theorem 4.1 to arrive at

$$|u(s)| \leq \epsilon\varrho\frac{(1+\psi)}{\delta\sigma} + \frac{\lambda^2}{\delta\sigma}\int_{t_0}^{t} \alpha(s)g(|u(s)|)ds + \frac{\lambda}{\delta\sigma}\int_{t_0}^{t} \rho(s)\gamma(|u(s)|)ds. \tag{64}$$

Using Theorem 3.3 to obtain

$$\begin{aligned} &|u(t)| \leq \epsilon\varrho\frac{(1+\psi)}{\delta\sigma}\Omega^{-1}\left(\Omega(1) + \frac{\lambda}{\delta\sigma}\int_{t_0}^{t} \rho(s)\gamma(F^{-1}(F(1)\right. \\ &\left.\left.\left. + \frac{\lambda^2}{\delta\sigma}\int_{t_0}^{s} \alpha(\xi)d\xi\right)\right)ds\right) F^{-1}\left(F(1) + \frac{\lambda^2}{\delta\sigma}\int_{t_0}^{t} \alpha(s)ds\right), \quad t \in \mathbf{I}. \end{aligned} \tag{65}$$

Using the conditions (x) and (xi) of Theorem 4.1 we have

$$|u(t)| \leq \epsilon\varrho\frac{(1+\psi)}{\delta\sigma}\Omega^{-1}\left(\Omega(1) + \frac{m\lambda}{\delta\sigma}\gamma\left(F^{-1}\left(F(1) + \frac{n\lambda^2}{\delta\sigma}\right)\right)\right) F^{-1}\left(F(1) + \frac{n\lambda^2}{\delta\sigma}\right). \tag{66}$$

Hence,

$$|u(t) - u(t_0)| \leq |u(t)| \leq \epsilon K_4,$$

where

$$K_4 = \varrho\frac{(1+\psi)}{\delta\sigma}\Omega^{-1}\left(\Omega(1) + \frac{m\lambda}{\delta\sigma}\gamma\left(F^{-1}\left(F(1) + \frac{n\lambda^2}{\delta\sigma}\right)\right)\right)F^{-1}\left(F(1) + \frac{n\lambda^2}{\delta\sigma}\right).$$

4.5 Theorem. Suppose the conditions of Theorems 4.2 and 4.3 emain valid. Then, the non-autonomous third order nonlinear differential equation (5) is Hyers-Ulam stable with Hyers-Ulam constant of equation (5) given as

$$K_5 = \varrho\frac{(1+\psi)}{\delta\sigma}\Omega^{-1}\left(\Omega(1) + \frac{n\lambda}{\delta\sigma}\right). \quad (67)$$

Proof. From inequality (12), it is clear that

$$-u'(t)\epsilon \leq u'''(t)u'(t) + \beta(t)f(u(t))u''(t)u'(t) + \rho(t)\gamma(u(t))u'(t) \leq u'(t)\epsilon. \quad (68)$$

Integrating from t_0 to t thrice, using Lemma3.2 and condition (i') of Theorem 4.3, we obtain

$$\int_{t_0}^{t} u'''(t)u'(s)ds + \int_{t_0}^{t} \beta(s)u''(s)\frac{d}{ds}U(u(s))ds + \int_{t_0}^{t} \rho(s)\gamma(u(s))u'(s)ds \leq \epsilon\int_{t_0}^{t} u'(s)ds. \quad (69)$$

The application of Theorem 3.2 implies there exists $\xi \in [t_0, t]$ such that

$$u'''(\xi)\int_{t_0}^{t} u'(s)ds + \int_{t_0}^{t} \beta(s)u''(s)\frac{d}{ds}U(u(s))ds + \int_{t_0}^{t} \rho(s)\gamma(u(s))u'(s)ds \leq \epsilon\int_{t_0}^{t} u'(s)ds. \quad (70)$$

By integration by parts and applying condition (iii') of Theorem 4.3 we get

$$u''(t)\sigma U(u(t)) \leq \epsilon\int_{t_0}^{t} |u'(s)|ds + |u'''(\xi)|\int_{t_0}^{t} |u'(s)|ds + |u'(t)|\int_{t_0}^{t} \rho(s)\gamma(u(s))ds. \quad (71)$$

Using conditions (i),(ii)(iii)and (iv) of Theorem 4.1 we obtain

$$|u(t)| \leq \epsilon\varrho\frac{(1+\psi)}{\delta\sigma} + \frac{\lambda}{\delta\sigma}\int_{t_0}^{t} \rho(s)\gamma(|u(s)|)ds \quad (72)$$

Using Lemma 3.1, we get

$$|u(t)| \leq \epsilon\varrho\frac{(1+\psi)}{\delta\sigma}\Omega^{-1}\left(\Omega(1) + \frac{\lambda}{\delta\sigma}\int_{t_0}^{t} \rho(s)ds\right), \quad t \in \mathbf{I}. \quad (73)$$

By condition (xi) of Theorem4.1 we arrive at

$$|u(t)| \leq \epsilon\varrho\frac{(1+\psi)}{\delta\sigma}\Omega^{-1}\left(\Omega(1) + \frac{n\lambda}{\delta\sigma}\right). \quad (74)$$

Hence,

$$|u(t) - u(t_0)| \leq |u(t)| \leq \epsilon K_5, \quad (75)$$

where

$$K_5 = \varrho\frac{(1+\psi)}{\delta\sigma}\Omega^{-1}\left(\Omega(1) + \frac{n\lambda}{\delta\sigma}\right).$$

4.4 Example Consider the Hyers-Ulam stability of the third order nonlinear differential equation

$$u'''(t) + t^{-4}u^4(t)u''(t) + t^{-2}u^2(t)u'(t) + t^{-6}u^2(t) = 0 \;\; t > 0$$

where$\beta(t) = t^{-4},\ f(u(t)) = u^4(t),\ \alpha(t) = t^{-2},\ g(u(t)) = u^2(t),\ \rho(t) = t^{-6},\ \gamma u(t) = u^2(t)$. By following the proof of the Theorem4.1, then the nonlinear differential equation is Hyers-Ulam stable.

4.6 Theorem. The non-autonomous third order nonlinear differential equation (6) has Hyers-Ulam stability, if the conditions of Theorems 2 and 11(`)@ remain valid. Then the Hyers-Ulam constant of equation (6) is given by

$$K_6 = \varrho \frac{(1+\psi)}{\xi} \Omega^{-1} \left(\Omega(1) + n \frac{\lambda^2}{\xi} \right) \tag{76}$$

Proof. From inequality (13) that

$$u'''(t) + \alpha(t) g(u(t)) u'(t) + \rho(t) r(u(t)) \le \epsilon. \tag{77}$$

Multiplying equation (77) by $u'(t)$, integrating from t_0 to t thrice and using Lemma 3.2 we obtain

$$\begin{aligned} &t^2 \int_{t_0}^{t} u'''(s) u'(s) ds + t^2 \int_{t_0}^{t} \alpha(t) g(u(s)) (u'(s))^2 ds \\ &+ t^2 \int_{t_0}^{t} \rho(s) \gamma(u(s)) u'(s) ds \le \epsilon t^2 \int_{t_0}^{t} u'(s) ds. \end{aligned} \tag{78}$$

Using conditions (i'), (iii') of Theorem 4.3 and integration by part swe obtain

$$\xi R(u(t)) \le \epsilon \int_{t_0}^{t} u'(s) ds - u'''(t) \int_{t_0}^{t} u'(s) ds - (u'(t))^2 \int_{t_0}^{t} \alpha(s) g(u(s)) ds. \tag{79}$$

Taking the absolute value of both sides, using conditions (ii), (iii), (iv) of the Theorem 4.1 and condition (ii') of the Theorem 4.3 we have

$$|u(t)|) \le \frac{(1+\psi)}{\xi} \varrho \epsilon + \frac{(\lambda)^2}{\xi} \int_{t_0}^{t} \alpha(s) g(|u(s)|) ds. \tag{80}$$

together with application of Lemma 3.1

$$|u(t)| \le \varrho \epsilon \frac{(1+\psi)}{\xi} \Omega^{-1} \left(\Omega(1) + \frac{\lambda^2}{\xi} \int_{t_0}^{t} \alpha(s) ds \right), \quad t \in \mathbf{I}. \tag{81}$$

By condition (xi) of Theorem 4.1 we have

$$|u(t)| \le \varrho \epsilon \frac{(1+\psi)}{\xi} \Omega^{-1} \left(\Omega(1) + n \frac{\lambda^2}{\xi} \right). \tag{82}$$

Hence,

$$|u(t) - u_0(t)| \le |u(t)| \le K_6 \epsilon.$$

Hyers-Ulam constant is given by

$$K_6 = \varrho \frac{(1+\psi)}{\xi} \Omega^{-1} \left(\Omega(1) + n \frac{\lambda^2}{\xi} \right).$$

CONCLUSION

This work on HUS results of third order nonlinear differential equations with nonlinear forcing term is very prominent in the stability of some problems such as hereditary, the surge in birth-rates, spreading of certain contagious diseases and so on. These problems appear directly in terms of integral equations and in terms of differential equations with certain conditions on the parameters of the model which can assist us to reduce the nonlinear equation to integral equations whereby GBB type inequality will be used to determine the stability of the solution of the nonlinear equation.

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References

[1] Aligaifiary H. Q. and Jung, S -M. : *On the Hyers-Ulam Stability of Differential Equations of Second Order*. Hindawi Publishing Cooperation Abstract and Appliedd Analysis Volume(2014),1-8.

[2] Alsina C. and Ger R. : On Some Inequalities and Stability Result Related to the Exponential Function. J. Inequl. Appl.**2**(1988), 373-38

[3] Bihari I. : *A generalisation of a Lemma of Bellman and its Application to Uniqueness Problem of Differential Equations*. -Acta Maths. Acad. Sc. Hung. **7**(1956), 71-94.

[4] Dhongade D.G. and Deo S.G. : *Some Generalisations of Bellman-Bihari Integral Inequalities*. Journal of Mathematical Analysis and Applications **44**(1973,)218-226.

[5] Fakunle I. and Arawomo P.O. : *Hyers-Ulam-Rassias stability of a Certain Perturbed Nonlinear Lienard Type Differential Equation.* Unilag Journal of Mathematics and Applications,Volume 3, (2023), 1-16.

[6] Fakunle.I and Arawomo P. O. : *Hyers-Ulam-Rassias stability of Nonlinear Second Order of A Perturbed Ordinary Differential Equation.* Proyecciones Journal of Mathematics. Vol 42,No 5, (2023),1157-1157, 2022.

[7] Fakunle I. and Arawomo, P. O. : *Hyers-Ulam Stability of a Perturbed Generalised Lienard Equation*. International Journal of Applied Mathematics.**32,No.3**(2019),479-489.

[8] Fakunle I. and Arawomo.P.O. : *Hyers-Ulam Stability of Certain Class of Nonlinear Second Order Differential Equations*. International Journal of Pure and Applied Mathematical Sciences.**11(1)**(2018),55-65.

[9] Fakunle I. and Arawomo,P.O. : *On Hyers-Ulam Stability of Nonlinear Second Order Ordinary and Functional Differential Equations*. International Journal of Differential Equations and Applications **17(1)**(2018)77-88.

[10] Gavruta P. Jung S-M. Li Y. : *Hyers-Ulam Stability for Second Order Liner Differential Equations with Boundary Conditions*.EJDE **80**(2011) pp1-7.

[11] Ince E.L. : *Ordinary differential Equation*. Messer.Longmans,Green and co. Heliopolis,pp42,(1926).

[12] Jung S-M. *Hyers-Ulam Stability of Linear Differential Equations of First Order*. Appl.Math.Lett.**17**(2004),1135-1140.

[13] Jung S-M. : : *Hyers-Ulam Stability of Linear Differential Equations of First Order*. Journal of Mathematics Analysis and Applications **33**(2005)139-146.

[14] Jung S-M. : *Hyers-Ulam Stability of Linear Differential Equations of First Order(II)* Appl.Math.Lett.**19**(2006)854-858.

[15] Li Y. : *Hyers-Ulam Stability of Linear Differential Equations of First Order*. Thai Journal of Mathematics 8(2) (2010)215-219.

[16] Li Y., Shen Y. : *Hyers-Ulam Stability of Nonhomogenous Linear Differential Equations of Second Order* International Journal of Mathematics and Mathematical Sciences (2009), Article ID576852, pp7.

[17] Obloza M. : *Hyers-Ulam Stability of the Linear Differential Equations*. Rocznik Nauk. Dydakt-Dydakt. Prac. Mat.**13**(1993), 259-270.

[18] Murray R. S.: *Schum's Outline of Theory and Problem of Calculus*, SI(Metric) Edition , International Edition 1974.

[19] Murali R. and Ponmana Selvan A. *Hyers-Ulam Stability of Third Order Linear Differential Equation*. Journal of Computer Mathematical Science.**9(10)**(2018),1334-1340.

[20] Murali R. and Ponmana Selvan A. : *Hyers-Ulam-Rassias Stability for the Linear Ordinary Differential Equation of Third Order*. Kragujevac Journal of Mathematics **42(4)** (2018),579-590.

[21] Miura T. ,Jung,S.-M. Takahasi,S.E. : *Hyers -Ulam-Rassias Stability of the Banach Space Valued Linear Differential Equations* $y' = \lambda y$ Korean Mathematics Society**41**(2004)99-1005.

[22] Qarawani M. N. : *Hyers-Ulam Stability of Linear and Nonlinear Differential Equations of Second Order*. Int. Journal of Applied Mathematical Research**1(4)**(2012), 422-432.

[23] Qarawani M. N.: *Hyers-Ulam Stability of a Generalised Second Order Nonlinear Differential Equations*. Applied Mathematics, **3**(2012)1857-1861.

[24] Rassias T. M.: *On the Stability of the Linear Mapping in Banach Spaces*. Proceedings of the American Mathematical Society, **72(2)**(1978),297-300

[25] Rus I.A. : *Ulam Stability of Ordinary Differential Equation* Studia Universities Babes-Bolyal Mathematical, **54(4)**(2010),306-309.

[26] Rus I. A. : *Ulam stability of Ordinary Differential Equations in a Banach Space*. Carpathian, J.Math.**26(1)**(2010)103-107.

[27] Tripathy A.K. and Satapathy A.: *Hyers-Ulam Stability of Third Order Euler's Differential Equations* Journal of Nonlinear Dynamics.(2014),Article I.D 4872577,6 pages

[28] Wang G., Zhou M. and L.Sun *Hyers-Ulam Stability of Linear Differential Equations of First Order* Appl.Math.Lett **21**(2008),1024-1028.