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SOLVABILITY ANALYSIS OF EQUALITY-CONSTRAINED OPTIMIZATION PROBLEMS WITH NONLINEAR FUNCTIONS

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ABSTRACT

This paper attempts fresh review techniques for solving constrained optimization problems with non-linear functions, emphasizing the Lagrange multipliers method. The paper reviews their solvability ability as employed in solving equality-constrained optimization problems. The first and second-order necessary conditions for obtaining optimal solutions are stated and discussed in this paper. The method of Lagrange multiplier was used to solve for optimal points. The work also demonstrates how the Borded Hessian Matrix can be used to determine the optimal solution to non-linear equality constraint problems with many constraints.

1. Introduction

The optimization problem can be seen as a problem of optimizing some functions relative to some set. The function of interest is called the objective function while the interest set is called the constraint set. It is the basis of establishing the best ways resources can be utilized to minimize cost and maximize profit. The optimization problem can be linear or Non-linear as the case may be and that results in linear programming (LP) and Non- linear programming (NLP) problems respectively. They are distinguished by the presence of linear or non-linear functions in either the objective function or constraints set and this problem can be unconstrained or constrained. These lead to different methods of solution.

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In 2019, authors in [1] proposed a new hybrid algorithm consisting of two meta-heuristic algorithms; Differential Evolution (DE) and Monarch Butterfly Optimization (MBO) and it showed that their algorithms give better results for the majority of the nonlinear systems and unconstrained optimization problems when compared with other existing algorithms in the literature. Researchers in [2] considered a nonlinear constrained optimization model for selecting a pipe route with a minimum length that considers seabed topography, obstacles, and pipe curvature requirements.

The authors in [3] also looked at a novel neural network-based model in a unified framework of zeroing neural networks (ZNN) to simultaneously solve multiple constraints for the time-dependent non-linear optimization.

In a work done by the authors in [4], they acknowledged the energy management systems in micro-grids using an optimization-based approach, optimizing the operating cost related to the energy purchased from the utility grid, the operation cost of the energy storage system, and revenue from the selling of energy to the utility grid. They used a constrained Particle Swarm Optimization-Based Model Predictive Control (CPSO-MPC) and a Linear Program-Based Optimization approach to solve the constrained optimization problem formulated in micro-grid energy management.

In 2016, my paper in [5] considered optimality conditions for equality-constrained optimization problems, and in 2024, we looked at the Linear Programming problem, using a case study of Vegas Restaurant and Bakery in Abakaliki, Ebonyi state. The paper attempted to solve the model for the Profit Maximization of the company using the Simplex Method, [6].

In [7] and [8], the authors applied the method of Newton Raphson's Iterative method Algorithms to solve some constrained optimization problems. However, these methods are so cumbersome when there are many constraint sets. This paper discussed the efficient method of solving the optimal points of a non-linear equality-constrained optimization problem.

2 AIM AND OBJECTIVES

The aim of this paper is to discuss the solvability of equality-constrained optimization problems involving nonlinear functions which is a fundamental aspect of mathematical optimization with broad applications in science and engineering. Understanding the conditions under which optimal solutions exist and can be efficiently determined is crucial for both theoretical advancements and practical implementations. To achieve this aim we consider the following objectives:

i. Investigate the Necessary Conditions for Optimality – Establishing the fundamental conditions that must be satisfied for a solution to be optimal in equality-constrained optimization problems involving nonlinear functions.
ii. Apply the Lagrange Multiplier Method – Demonstrating how the method of Lagrange multipliers can be used to identify optimal points in nonlinear optimization problems with equality constraints.
iii. Utilize the Bordered Hessian Matrix – Exploring the role of the bordered Hessian matrix in verifying the optimality of solutions, particularly in cases involving multiple constraints.

3 METHODOLOGY

In mathematical optimization, constrained optimization is a process of optimizing an objective function concerning some variables in the presence of constraints on those variables. The objective function is either a cost function or energy function which is to be minimized, or a reward function or utility function, which is to be maximized while constraint sets are functions.

For constrained and unconstrained functions, optimization theory uses differential calculus to obtain minimum and maximum points. This paper uses the Lagrange multiplier method to solve equality-constrained problems with Non-linear Functions.

3.1. Equality constraints optimization problem

The equality constraints optimization problem is of the form
Optimize $A(x)$ $(P1)$
Subject to $x \epsilon K \neq \emptyset$
Where $K = P \cap \{x \in \mathbb{R}^n : g(x) = 0\}$, P is an open subset of $\mathbb{R}^n$, $A: K \subset \mathbb{R}^n \rightarrow \mathbb{R}$

3.2. Lagrange multipliers method

The method of Lagrange's multipliers is an interesting technique for finding the maximum or minimum of a function of the form $A(x, y, z)$ subject to equality constraints of the form $g(x, y, z) = b$ or $g(x, y, z) = 0$.
The method of the lagrangian multiplier is based on the theorem of Lagrange.
The theorem of Lagrange makes sense in characterizing optima of equality-constrained optimization problems in terms of objective function behavior and the constraint functions 'g' at these points. We define some terms that will help us present the first necessary optimality condition called Lagrange Theorem.

Definition 3.1: In a matrix form, a point x^* is a regular point of the equality constraint problem if the Jacobian matrix of g at x^*, denoted by

Definition 3.2: $J\,(g(x^*) = \left[\frac{\partial g_i}{\partial x_j}\right], i = 1, \ldots, m, j = 1, \ldots, n$ Jacobian matrix has a full rank (that is rank m), where $g = (g_1, g_2, \ldots, g_m)$. This means t, the rank of the Jacobian matrix of g at x^* is equal to the number of constraints.

Definition 3.3: The constraint qualification under equality constraints is the condition in the Lagrangian theorem that, $R\,\nabla g(x^*) = m$, (number of constraints). It ensures that $\nabla g(x^*)$ contains an invertible $m \times m$ submatrix which may be used to define the vector λ^*. $R\,\nabla g(x^*)$, denotes the rank of the constraints

This means that if this constraint qualification is not meet, then, the conclusion of the theorem will also fail, that is, if x^* is a local optimum for which, $R\,\nabla g(x^*) < m$, then, there will not be a vector λ^* such that the set of all critical point of Lagrange function, L that contains the set of all local maxima and minima of A on K at which the constraint qualification is met.
The consequence of this property is that, if a global minimizer, x^* of the function, A exists to the given problem $(P1)$ and the constraint qualification is met at x^*. Then there exists λ^* such that(x^*, λ^*) is a critical point of L then, we say that a pair (x^*, λ^*) meets the first order conditions of the equality constrained optimization problem if it satisfies $g(x^*) = 0$ and $\nabla A(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla g_i\,(x^*) = 0$.
The function associated with the equality-constrained optimization problem is called the Lagrangian function and is defined as follows:

$L: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$,

$$L\,(x^*, \lambda^*) = A(x^*) + \sum_{i=1}^{m} \lambda_i{}^* g_i(x^*) = 0\,, \forall\, x \in \mathbb{R}^n\,, \lambda \in \mathbb{R}^m.$$

With this L −function, we get the equations as follows:

$$\frac{\partial L(x, \lambda)}{\partial x_j} = 0, j = 1, \ldots, n$$

$$\frac{\partial L(x, \lambda)}{\partial \lambda_i} = 0, i = 1, \ldots, m$$

Step 1: Construct the Lagrange function
Introduce a new variable, λ, and define a new function L as follows:

$$L(X, \lambda) = A(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$

This function L is called the "Lagrange function" and the new variable λ is referred to as a "Lagrange multiplier"
Step 2: Critical points
Take the gradient of the Lagrange and set each component to zero(0).
Step 3: Check for optimality
Solve for each of the variables by solving the equations formed simultaneously. Remove the λ_0 component, then plug it into the function, A, since A does not have λ as an input. Whichever one gives the greatest (or smallest) value is the maximum (or minimum) point you are seeking

3.3 BORDED HESSIAN MATRIX METHOD

Here, we discuss the method of solution when the candidates for optimal solution are many that it involves a lot of computations. This method is called Borded Hessian Matrix dented by H^B and it uses the second partial of the L −function concerning the decision variables and the first partial of the constraints functions.

$$H^B = \left[\begin{array}{c|c} 0 & P \\ \hline P^T & Q \end{array}\right]_{(m+n)\times(m+n)}$$

Where $0 = n \times n$, zero matrix, $P = \begin{bmatrix} \nabla \mathrm{g}_1(\mathrm{x}) \\ . \\ . \\ . \\ \nabla \mathrm{g}_m(\mathrm{x}) \end{bmatrix}_{m\times n}$

And
$$Q = \left[\frac{\partial^2 \mathrm{L}(\mathrm{x},\lambda)}{\partial \mathrm{x}_i \mathrm{x}_j}\right]_{n\times n}, \forall\, i, j$$

Here, if the critical points $(x^*, \lambda^*)$are compiled for the Lagrangian function $L(x^*, \lambda^*)$ and $H^B$is evaluated at (x^*, λ^*)then x^* is:

i. A minimum point if starting with the principal minor determinant of order $(2m+1)$, he last$(n-m)$ principal minor determinants of H^B has the sign of$(-1)^m$

ii. A maximum point if starting with the principal major determinants of order $(2m+1)$, the last $(n-m)$ principal minor determinants of H^B form an alternating sign pattern with $(-1)^{m+1}$

To solve constrained optimization problems with non-linear functions using the Lagrangian multiplier method the following steps are necessary:

- Construct the Lagrangian function.
- Critical points
- Check for optimality

4 APPLICATIONS

Here, we try to present the first-necessary and second-order necessary conditions that guarantee the optimum solution for the equality-constrained problem and use constructed examples to verify its effectiveness.

4.1. First Order Conditions for Equality Constrained problem.

The main result here is the Theorem of Lagrange and is regarded as the first-order necessary optimality conditions for the equality-constrained optimization problem.

Theorem 4.1. Let $A : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous first differentiable, C^1 functions, $i = 1, \ldots, m$, suppose x^* is a local optimum of A on the set
$K = P \cap \{x \in \mathbb{R}^n : g_i(x) = 0, i = 1, \ldots, \text{m}\}, p \subset \mathbb{R}^n$, is open.
Suppose also that $R(\nabla g(x^*)) = m$, then, there exists a vector
$\lambda^* = ({\lambda_1}^*, {\lambda_2}^*, \ldots, {\lambda_m}^*) \in \mathbb{R}^m,\ \ni\ \nabla f(x^*) + \sum_{i=1}^{m} {\lambda_i}^* \nabla g_i\,(x^*) = 0,$
Where ${\lambda_i}^{*'}s$ are called the Lagrangian multipliers associated with the local optimum x^* and $R\nabla g\,(x^*)$ is the rank of $\nabla g\,(x^*) = \frac{\partial g_i}{\partial x_j}\,, j = 1, \ldots, n\,, i = 1, \ldots, m.$

We verify this theorem using a constructed example:
Consider the problem, with the function, $A: \mathbb{R}^2 \to \mathbb{R}$
Minimize$A(x) = {x_1}^2 + 3{x_2}^2$
Subject to ${x_1}^2 + {x_2}^2 = 1$ [6]
The constraint equation reduces to $g(x) = 0$ and, $K = P \cap \{x \in \mathbb{R}^2 : g(x) = 0\}$ and write the above problem as,
Minimize$A(x) = {x_1}^2 + 3{x_2}^2$
Subject to, $K = P \cap \{x \in \mathbb{R}^2 : g(x) = 0\}$.
Let, $P \subset \mathbb{R}^2$, then $K = \{x \in \mathbb{R}^2 : {x_1}^2 + {x_2}^2 = 1\}$.
Here, the objection function, $A(x)$ is a nonlinear function and the constraint, K is also a non-linear function.
Now, we verify the Lagrange theorem.
Applying the Lagrange multiplier method, we construct the Lagrange function,
$L(\bar{x}, \lambda) = A(x) + \sum_{i=1}^{m} \lambda_i g_i\,(\bar{x}) = A(\bar{x}) + \lambda_1 g_1(\bar{x}), \forall (x, \lambda) \in K \times \mathbb{R}$
$L(\bar{x}, \lambda) = {x_1}^2 + 3{x_2}^2 + \lambda({x_1}^2 + {x_2}^2 - 1)$
We show, that the condition existence of the global minimizer, and the constraint qualification are met.
First, since A is a continuous function on K and K is a compact set, thus by the existence theorem, there exists a global minimizer of A in K. This implies that the critical points of the Lagrange function L will, contain the set of global minimizers.

Second, we check for the constraint qualification. The derivative of the constraint function g at any point is given by $(x_1^*, x_2^*) \in \mathbb{R}^2$.
Given by
$\nabla g(x_1^*, x_2^*) = (2x_1, 2x_2)$,$\nabla g(x) = 0 \Rightarrow 2\lambda x_1 = 0$,$or$ $2\lambda x_2 = 0$, here, x_1^* and x_2^* cannot be zero simultaneously on K otherwise, ${x_1}^2 + {x_2}^2 \neq 1$. This implies that we must have $R\big(\nabla g(x_1^*, x_2^*)\big) = 1$ at $(x_1^*, x_2^*) \in K$. Therefore, the constraint qualification is satisfied everywhere on K
Since, $R\big(\nabla g(x_1^*, x_2^*)\big) = m = 1$,
Then, we can find $\lambda^* \in \mathbb{R}$ such that for x^* a minimizer, we have;
$\nabla A(x_1^*, x_2^*) + \lambda^* \nabla g(x_1^*, x_2^*) = 0$ and, $\big(g(x_1^*, x_2^*)\big) = 0$.
The Lagrange equation reduces to:

$$\frac{\partial L}{\partial x_1}(x_1^*, x_2^*, \lambda^*) = 2x_1 + 2\lambda x_1 = 0 \quad (1)$$

$$\frac{\partial L}{\partial x_2}(x_1^*, x_2^*, \lambda^*) = 6x_2 + 2\lambda x_2 = 0 \quad (2)$$

$$\frac{\partial L}{\partial \lambda}(x_1^*, x_2^*, \lambda^*) = {x_1}^2 + {x_2}^2 - 1 \quad (3)$$

From, (1), $2x_1^*(1+\lambda) = 0$
From,(2), $2x_2^*(3+\lambda) = 0$
Therefore, $({x_1}^*, {x_2}^*, \lambda^*) \in \mathbb{R}^3$ is the critical points of L
Hence, the candidates for minimizer are:

$$({x_1}^*, {x_2}^*, \lambda^*) = \begin{cases} (0,1,-3) \\ (0,-1,-3) \\ (1,0,-1) \\ (-1,0,-1) \end{cases}$$

If we compute the $A({x_1}^*, {x_2}^*)$ at these four points, we observe that the points $(1,0)$ and $(-1,0)$ are the global minimizers of A on K while the points $(0,1)$ and $(0,-1)$ are the global maximizers of A on K.

4.2 Second-Order Necessary Optimality Conditions of Lagrange

Theorem 4.2. Let $A : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}^k$ be a twice continuous differentiable, C^2 functions, $i = 1, \ldots, k$, suppose there $x^* \in \mathbb{R}^n$, and $\lambda^* \in \mathbb{R}^k$ such that $R(\nabla g(x^*)) = k$ and $\nabla f(x^*) + \sum_{i=1}^{m} {\lambda_i}^* \nabla g_i(x^*) = 0$,
Define $Z(x^*) = \{z \in \mathbb{R}^n : \nabla g_i(x^*) z = 0, i = 1, \ldots, \mathrm{m}\}$, and let
$\nabla^2 L(x^*, \lambda^*) = \nabla^2 A(x^*) + \sum_{i=1}^{m} {\lambda_i}^* \nabla^2 g_i(x^*)$, denote the $n \times n$ symmetric matrix of $L(x^*, \lambda^*)$, if A has a local minimum on Katx^*, then $z^2 \nabla^2 g_i(x^*) z \geq 0$ for all
z $\in z(x^*)$ and, if A has a local maximum on Katx^*, then $z^2 \nabla^2 g_i(x^*) z \leq 0$ for all
z $\in z(x^*)$. [6]
Observe that, Theorem 4.2 above theorem can be characterized in terms of the definiteness of a symmetric $n \times n$ matrix Q on the set $\{z \neq 0 : Bz = 0\}$, where B is an $m \times n$ matrix of rank m. Because of this characterization, we use an alternative way to verify Theorem 4.2. The alternative way is called the Borded Hessian matrix denoted byH^B, which is discussed in section 3.2
Here, we apply this matrix in some examples to check for the optima of the solution.
Example 3.1
Consider the problem
Minimize$f(x) = 2{x_1}^2 + {x_2}^2 + 3{x_3}^2 + 10x_1 + 8x_2 + 6x_3 - 100$
Subject to $x_1 + x_2 + x_3 = 20$
$x_1, x_2, x_3 \geq 0$
Solution

We construct the Lagrange function;

$$L(X,\lambda) = A(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$
$$= 2{x_1}^2 + {x_2}^2 + 3{x_3}^2 + 10x_1 + 8x_2 + 6x_3 - 100 + \lambda(x_1 + x_2 + x_3 - 20$$

The necessary conditions for a stationary point are:

$$\frac{\partial L}{\partial x_1} = 4x_1 + 10 + \lambda = 0 \qquad (i)$$
$$\frac{\partial L}{\partial x_2} = 2x_2 + 8 + \lambda = 0 \qquad (ii)$$
$$\frac{\partial L}{\partial x_3} = 6x_3 + 6 + \lambda = 0 \qquad (iii)$$
$$\frac{\partial L}{\partial \lambda} = x_1 + x_2 + x_3 - 20 = 0 \qquad (iv)$$

Solving $(i), (ii), (iii)$ and (iv) simultaneously,

From(i); $x_1 = \frac{-(10+\lambda)}{4}$ $\qquad (v)$

From (ii); $x_2 = \frac{-(8+\lambda)}{2}$ $\qquad (vi)$

From (iii); $x_3 = \frac{-(6+\lambda)}{6}$ $\qquad (vii)$

Put $(v), (vi)$ and (vii) in (iv),

$$\Rightarrow \frac{-(10+\lambda)}{4} - \frac{(8+\lambda)}{2} - \frac{(6+\lambda)}{6} - 20 = 0$$

$$\frac{-30 - 3\lambda - 48 - 6\lambda - 12 - 2\lambda - 240}{12} = 0$$

$$-11\lambda - 330 = 0$$

$$11\lambda = -330$$

$$\Rightarrow \lambda = -30$$

Put $\lambda = -30$ in $(v), (vi)$ and (vii);

$$\Rightarrow x_1 = \frac{-(10-30)}{4} = 5 \quad x_2 = \frac{-(8-30)}{2} = 11$$

$$x_3 = \frac{-(6-30)}{6} = 4 \Rightarrow x^* = (5,11,4), \lambda^* = -30$$

Therefore, $({x_1}^*, {x_2}^*, x_3,^* \lambda^*) = (5,11,4-30)$

To determine whether the point, $({x_1}^*, {x_2}^*, {x_3}^*) = (5,11,4)$ is the solution is optimal, we construct the H^B;

$$H^B = \begin{bmatrix} 0 & P \\ P^T & Q \end{bmatrix}, \text{ where, } \qquad P = \begin{bmatrix} \nabla g_1(x) \\ \nabla g_2(x) \\ \nabla g_3(x) \end{bmatrix} = \begin{bmatrix} \frac{\partial g}{\partial x_1} \\ \frac{\partial g}{\partial x_2} \\ \frac{\partial g}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and $Q = \left[\frac{\partial^2 \text{L}(\text{x},\lambda)}{\partial \text{x}_i \text{x}_j}\right]_{3\times 3}$, $\forall\ i = 1,2,3;\ j = 1,2,3$

$$Q = \begin{bmatrix} \frac{\partial^2 L}{\partial {x_1}^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial {x_2}^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial {x_3}^2} \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\Rightarrow H^B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 6 \end{bmatrix}$$

Let $n =$ number of variables $= 3$
$m =$ number of constraints $= 1,$
Checking the principal minor determinant of H^B of order $(2m + 1) = (2 + 1) = 3$

$$\text{Let } \Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 6 \end{vmatrix} = -6$$

$$\Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 6 \end{vmatrix} = -44$$

Since Δ_3 and Δ_4 are both negative and have the sign of $(-1)^m$, then from the condition (i) in section 3.2 , $x^* = (5,11,4)$ is a minimum point, and $A(x^*) = A(5,11,4) = 281.$
Hence, the condition in theorem 4.2 is verified since H^B is a positive semi definite.

Example 3.2

Consider this problem, now with more than one constraint:
Optimize $Z = 4{x_1}^2 + 2{x_2}^2 + {x_3}^2 - 4x_1x_2$
Subject to $x_1 + x_2 + x_3 = 15$
$2x_1 - x_2 + 2x_3 = 20$
$x_1, x_2, x_3 \geq 0$

Solution

We construct the Lagrange function;

$$L(X,\lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i\,(x)$$
$$= 4{x_1}^2 + 2{x_2}^2 + {x_3}^2 - 4x_1x_2 + \lambda_1(x_1 + x_2 + x_3 - 15) + \lambda_2(2x_1 - x_2 + 2x_3 - 20)$$

The necessary conditions for stationary point are:

$$\frac{\partial L}{\partial x_1} = 8x_1 - 4x_2 + \lambda_1 + 2\lambda_2 = 0 \qquad (i)$$

$$\frac{\partial L}{\partial x_2} = 4x_2 - 4x_1 + \lambda_1 - \lambda_2 = 0 \qquad (ii)$$

$$\frac{\partial L}{\partial x_3} = 2x_3 + \lambda_1 + 2\lambda_2 = 0 \qquad (iii)$$

$$\frac{\partial L}{\partial \lambda_1} = x_1 + x_2 + x_3 - 15 = 0 \qquad (iv)$$

$$\frac{\partial L}{\partial \lambda_2} = 2x_1 - x_2 + 2x_3 - 20 = 0 \qquad (v)$$

Solving $(i),(ii),(iii),(iv)$ and (v) simultaneously,
From (iii); $\lambda_1 + 2\lambda_2 = -2x_3$ $\quad (vi)$
Put (vi) in (i); $8x_1 - 4x_2 - 2x_3 = 0$ $\quad (vii)$
From (iv); $x_1 + x_3 = 15 - x_2$
From (v); $2(x_1 + x_3) - x_2 - 20 = 0$

$$\Rightarrow 2(15 - x_2) - x_2 - 20 = 0$$

$$30 - 2x_2 - x_2 - 20 = 0$$

$10 - 3x_2 = 0, x_2 = \frac{10}{3}$

Put $x_2 = \frac{10}{3}$ in (vii); $8x_1 - 4\left(\frac{10}{3}\right) - 2x_3 = 0$

Multiplying through by 3; $24x_1 - 40 - 6x_3 = 0$

$$\Rightarrow 24x_1 - 6x_3 = 40 \qquad (viii)$$

Put $x_2 = \frac{10}{3}$ in (iv); $x_1 + \frac{10}{3} + x_3 - 15 = 0$

Multiplying through by 3; $3x_1 + 10 + 3x_3 - 45 = 0$

$$\Rightarrow 3x_1 + 3x_3 = 35$$

Solving $(viii)$ and (ix) simultaneously,
$(viii) \times 3$; $\quad 72x_1 - 18x_3 = 120 \qquad \cdots (x)$
$(ix) \times -6$; $\quad -18x_1 + -18x_3 = -210 \qquad \cdots (xi)$
Subtract (xi) from (x); $\quad 90x_1 = 330$

$$x_1 = \frac{11}{3}$$

Put $x_1 = \frac{11}{3}$ and $x_2 = \frac{10}{3}$ in (ii); $4\left(\frac{10}{3}\right) - 4\left(\frac{11}{3}\right) + \lambda_1 - \lambda_2 = 0$

$$\lambda_1 - \lambda_2 = 4\left(\frac{11}{3}\right) - 4\left(\frac{10}{3}\right)$$

$$\lambda_1 - \lambda_2 = \frac{44}{3} - \frac{40}{3}$$

$$\lambda_1 - \lambda_2 = \frac{4}{3} \qquad \cdots (xii)$$

Also put $x_3 = 8$ in (vi); $\lambda_1 + 2\lambda_2 = -2(8)$

$$\lambda_1 + 2\lambda_2 = -16 \qquad \cdots (xiii)$$

Solving (xii) and $(xiii)$simultaneously,
Subtract (xii) from $(xiii)$; $3\lambda_2 = \frac{-52}{3}$

$$\lambda_2 = \frac{-52}{9}$$

Put $\lambda_2 = \frac{-52}{9}$ in (xii); $\lambda_1 + \frac{52}{9} = \frac{4}{3}$

$$\lambda_1 = \frac{4}{3} - \frac{52}{9}$$

$$\lambda_1 = \frac{12-52}{9}$$
$$\lambda_1 = \frac{-40}{9}$$

$\Rightarrow x^* = (x_1{}^*, x_2{}^*, x_3{}^*) = \left(\frac{11}{3}, \frac{10}{3}, 8\right)$

and $\lambda^* = (\lambda_1{}^*, \lambda_2{}^*) = \left(\frac{-40}{9}, \frac{-52}{9}\right)$

To determine whether the point $(x_1{}^*, x_2{}^*, x_3{}^*) = \left(\frac{11}{3}, \frac{10}{3}, 8\right)$ is maximum or minimum, we construct the Borded Hessian matrix, H^B;

$$H^B = \begin{bmatrix} 0 & P \\ P^T & Q \end{bmatrix}$$

where , $P = \begin{bmatrix} \nabla g_1(x) \\ \nabla g_2(x) \\ \nabla g_3(x) \end{bmatrix} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \end{bmatrix}$

and $Q = \left[\frac{\partial^2 L(x,\lambda)}{\partial x_i x_j}\right]_{3\times 3}$, $\forall\ i = 1,2,3;\ j = 1,2,3$

$$Q = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1{}^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2{}^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3{}^2} \end{bmatrix} = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow H^B = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ 1 & 2 & 8 & -4 & 0 \\ 1 & -1 & -4 & 4 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{bmatrix}$$

Let $n =$ number of variables $= 3$

$m =$ number of constraints $= 2$,

Checking the principal minor determinant of H^B starting from order $(2m + 1) = (2(2) + 1) = 5$

$$|H^B| = \begin{vmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ 1 & 2 & 8 & -4 & 0 \\ 1 & -1 & -4 & 4 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{vmatrix}$$

To find the determinant, we transform H^B into a triangular matrix using row reduction, noting that the signs beside the matrices below changes when a row exchange is made, that is, doing row exchanges changes the sign of the determinant.

Performing the operations, where $R(O) =$ old row

$R(N) =$ new row:

$$R_1(O) \leftrightarrow R_3(O): \begin{vmatrix} 1 & 2 & 8 & -4 & 0 \\ 0 & 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & -1 & -4 & 4 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{vmatrix} (-),$$

$$R_4(N) \to R_4(O) - R_1(O): \begin{vmatrix} 1 & 2 & 8 & -4 & 0 \\ 0 & 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & -3 & -12 & 8 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{vmatrix} (-)$$

$$R_5(N) \to R_5(O) - R_1(O): \begin{vmatrix} 1 & 2 & 8 & -4 & 0 \\ 0 & 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & -3 & -12 & 8 & 0 \\ 0 & 0 & -8 & 4 & 2 \end{vmatrix} (-)$$

$$R_2(O) \leftrightarrow R_4(O): \begin{vmatrix} 1 & 2 & 8 & -4 & 0 \\ 0 & -3 & -12 & 8 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ 0 & 0 & -8 & 4 & 2 \end{vmatrix} (+)$$

$$\begin{matrix} R_4(N) \to R_4(O) - 2R_3(O) \\ R_5(N) \to R_5(O) + 8R_3(O) \end{matrix}: \begin{vmatrix} 1 & 2 & 8 & -4 & 0 \\ 0 & -3 & -12 & 8 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ 0 & 0 & -8 & 4 & 2 \end{vmatrix} (+)$$

Now it is an upper triangular matrix, multiply the entries in the leading diagonal.

$$\Rightarrow |H^B| = 1 \times (-3) \times 1 \times (-3) \times 10$$
$$= 90 > 0$$

Since $|H^B|$ has the sign of $(-1)^m$, then from the condition (i) in section 3.2.

Hence, $(x_1{}^*, x_2{}^*, x_3{}^*) = \left(\frac{11}{3}, \frac{10}{3}, 8\right)$ is a minimum point, and the minimum value, $A(x)^* = A\left(\frac{11}{3}, \frac{10}{3}, 8\right) = \frac{820}{9}$.

CONCLUSION

In conclusion, this paper has been able to present the first and second-order necessary conditions, a non-linear equality constraint problem must satisfy to have an optimal solution. The paper applied the Lagrange method to solve for optimal points of equality constraint problem in a constructed example. The work also demonstrated how the Borded matrix can be used to determine the optimal point.

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