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A STUDY OF MATHEMATICAL MODELLING OF CONTROL SYSTEMS: (A CASE STUDY OF THE INVERTED PENDULUM)

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ABSTRACT

This paper aims to study the mathematical modeling of control systems by examining the dynamics of an inverted pendulum and analyzing the system of nonlinear differential equations associated with it. The objective is to formulate a mathematical model of the inverted pendulum system. The Euler-Lagrange equations were utilized to derive the equations of motion. Additionally, the paper seeks to compute the controllability of the linearized system, obtain the transfer functions, and analyze the state-space representation of the model. Finally, graphical profiles illustrating the system's behavior are provided. This work stabilized the pendulum, enabling precise and rapid control of the carriage's position on the track

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1. Introduction

System theory is a trans-disciplinary study of systems and deals with making decisions because of the uncertainty in both mechanical and human systems. A major challenge is the case where the current outcome(s)/output(s) of a system depends on the history of the control inputs. For example, when trying to maintain a submarine in motion to be at a constant or desirable depth below the ocean surface. Here, the main desired output is the submarine depth below the ocean surface, and it depends largely upon the submarine stern plane, the bow, and the position of the submarine past control surfaces. Developing any theory for design or analysis usually entails the abstraction of reality by using approximates and as well utilizing mathematical relations.

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In cases of controlling dynamic systems, these mathematical relations usually take the form of simple or complex equations, ordinary or partial differential equations, and linear or nonlinear equations. The main system variable of interest is called the state variable, whereas the variables that can be manually or automatically manipulated are usually called control variables.
The system control engineer has to deal with the issues of uncertainty and also complexities associated with multi-variable dynamic systems which are described using complex differential equations.

In [1], the authors considered the rigid finite-dimensional models which are described by ordinary differential equations (ODEs), and, derived a Mathematical model using the Hamiltonian principle and variational methods, which were formulated by the coupling of partial differential equations (PDE) and ODE. In (2014), the researchers in [2] used linear quadratic regulator (LQR) and proportional-integral-derivative (PID) control methods to control the nonlinear dynamical system. They actually, modeled and simulated the optimal control design of a nonlinear inverted pendulum-cart dynamic system using PID controller and LQR methods.

Fractional calculus was used by the authors in [3] to design a robust fractional-order PID (PIλDμ) controller for stabilization and tracking control of an inverted pendulum (IP) system. In their work, they used a particle swarm optimization (PSO) based direct tuning technique to design two PIλDμ controllers for the IP system without linearizing the actual nonlinear model. In 2019, the researchers in [4] in their paper titled Modeling and Analysis of an inverted pendulum used Newton-Euler formulation and Lagrange-Euler formulation to obtain Equations of motion (EOM) for the inverted pendulum system. They mounted an inverted pendulum on a cart and the dynamic behavior of the modeled system is analyzed through simulation results.

The researchers present an overview of the IP control system augmented by a comparative analysis of multiple control strategies in [5]. They studied and analyzed the approaches based on several parameters. The researchers used Non-linear techniques and AI-based approaches to mitigate IP nonlinearity and stabilize its unbalanced form.

In this paper, we attempted to study the mathematical modeling of control systems by examining the dynamics of an inverted pendulum and analyzing the system of nonlinear differential equations associated with it. The work utilized the Euler-Lagrange equations to derive the equations of motion. The reasons for choosing the Inverted Pendulum as the system to be modeled include; (i) It is the most easily available system for laboratory usage and, (ii) It is a nonlinear system, which can be treated as linear, without much error, for quite a wide range of variation.

1.1 Aim and Objectives of the Study

This study aims to study the Mathematical Modeling of Control Systems by considering the dynamics of an inverted pendulum and analyzing the system of nonlinear differential equations associated with an inverted pendulum, while the objectives of the study include:

i. Formulate the mathematical model of the Inverted Pendulum System.
ii. Compute the Controllability of the linearized system.
iii. Apply the Laplace transform to obtain the Transfer Functions.
iv. Obtain and analyze the state space form of the model.
v. Provide the necessary graphical profiles describing the system’s behavior

1.2 Statement of the problem

This research is about the mathematical modeling and control of an inverted pendulum. In a most basic manner, this is like trying to balance a rod on one's finger, though cases of the rod are placed on top of a moving cart which can move left or right to keep the inverted pendulum upright. This is a classical or conventional control problem about an inherently unstable system. The concept is that the pendulum, which is pivotally hinged to the cart, stays upright and vertically inclined while the cart moves.

The inverted pendulum is a mechanical system which is usually unstable. Our study is mainly geared toward controlling the inverted pendulum and, stabilizing it.

2 METHODOLOGY

The method used in this work is **the Euler-Lagrange (EL) Method, generating The Euler-Lagrange (EL) equations.** The EL-equations which involves the Lagrangian of the system, is a significant mathematical tool for obtaining the equations of motion of any dynamical system. The state-space formulas and transfer functions can be found using these equations. The system's potential and kinetic energies were used to express the generic formula for the Lagrangian (L). Thus;

$$L = K - V \tag{1}$$

Where, K = Kinetic Energy (KE), V = Potential Energy (PE).

$$K = \tfrac{1}{2}Mv_m^2 + \tfrac{1}{2}mv_m^2 + \tfrac{1}{2}I\omega^2 \tag{2}$$

$$\omega = \dot{\theta}, \quad v_m^2 = \dot{x}_m^2 + \dot{y}_m^2 \tag{3}$$

$$x_m = x - l\sin\theta \;\Rightarrow\; \dot{x}_m = \dot{x} - l\dot{\theta}\cos\theta \tag{4}$$

$$y_m = l\cos\theta \;\Rightarrow\; \dot{y}_m = -l\dot{\theta}\sin\theta \tag{5}$$

2.1 Kinetic Energy

The general formula to determine K becomes;

$$\begin{aligned} K &= \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\left[\tfrac{d}{dt}\left(x - l\sin\theta\right)^2\right] + \tfrac{1}{2}m\left[\tfrac{d}{dt}\left(-l\sin\theta\right)^2\right] + \tfrac{1}{2}I\dot{\theta}^2 \\ &= \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\left[\left(\dot{x} - l\dot{\theta}\cos\theta\right)^2 + \left(-l\dot{\theta}\sin\theta\right)^2\right] + \tfrac{1}{2}I\dot{\theta}^2 \\ &= \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\left(\dot{x}^2 - 2l\dot{\theta}\dot{x}\cos\theta + l^2\dot{\theta}^2\cos^2\theta + l^2\dot{\theta}^2\sin^2\theta\right) + \tfrac{1}{2}I\dot{\theta}^2 \\ \Rightarrow \quad K &= \tfrac{1}{2}\left(M + m\right)\dot{x}^2 + \tfrac{1}{2}ml^2\dot{\theta}^2 - ml\dot{\theta}\dot{x}\cos\theta + \tfrac{1}{2}I\dot{\theta}^2 \end{aligned} \tag{6}$$

2.2 Potential Energy

The potential energy corresponding to the system is expressed by

$$V = mgh = mgl\cos\theta \tag{7}$$

3 RESULTS

In control engineering, the inverted pendulum (IP) is regarded as one of the most complex systems to manage. It has become a popular topic for mathematical modeling due to its importance in this sector, with attempts concentrated on analyzing its model and creating a linear compensator based on the PID control law. Being an unstable system, an Inverted Pendulum is an important control problem in Control System Engineering, and the main interest is in controlling its dynamics.

3.1 APPLICATIONS OF INVERTED PENDULUM

Some applications of inverted pendulum (IP):

3.1.1 Simulation of dynamics of a robotic arm

The control mechanisms of robotic arms are comparable to the inverted pendulum problem. When the Centre of pressure is placed below the Centre of gravity of the arm, the Inverted Pendulum's dynamics resemble those of a robotic arm, leading to instability. The robotic arm behaves similarly to an inverted pendulum in these circumstances.

3.1.2. Model of a human standing still

Maintaining balance while standing erect is crucial for everyday activities. Muscles are activated to maintain balance by the central nervous system (CNS), which keeps an eye on posture and any changes in it. The inverted pendulum is commonly recognized as an effective model for a person who is motionless.

Feedback on its condition is crucial for stabilization because an inverted pendulum (without springs connected) is by nature unstable. Two primary models are often used to describe CNS feedback control:
Time-invariant, linear feedback control.
Linear feedback is outside a defined threshold, with no sensory feedback within the threshold.
Additionally, passive mechanisms like muscle stiffness and supportive tissues, which can be represented as a combination of spring and damper systems, contribute to maintaining balance.

3.1.3 Problem Definition

Balancing a pendulum in the inverted position is nearly impossible without the application of a third-party force. Providing this degree of force to the pendulum carriage is made possible by the Carriage Balanced Inverted Pendulum (CBIP) system, which is depicted in Figure 10. In this system, a DC servo motor, connected through a belt drive mechanism, provides the required control force. The CBIP system can output various parameters, including carriage position, velocity, pendulum angle, and angular velocity; however, only the pendulum angle is considered in this case. This angle is fed into an Analog Controller that regulates the servo motor, ensuring steady and continuous traction.

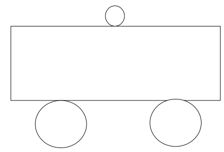


Fig. 1 Carriage Balanced Inverted Pendulum

The study aims to stabilize a pendulum by making sure that the carriage position on the track can be swiftly and precisely regulated while keeping the pendulum in a stable inverted posture while moving. The system consists of a cart that can move forward and backward, with a pendulum attached at its base, allowing it to swing in the same plane as the cart. This setup enables the pendulum to freely fall along the cart's axis of motion. The objective is to control the system such that the pendulum stays balanced and upright, even when subjected to sudden disturbances. If the pendulum begins off-center, it tends to fall, prompting the cart to move in the opposite direction to counterbalance it. Similarly, moving the cart can cause the pendulum to deviate from its upright position. This interdependence between the cart and pendulum makes the control system more complex than it initially appeared. Due to these intricacies, this problem is often used to illustrate the principles of fuzzy control systems.The inverted pendulum cart moves along a track and is driven by a belt connected to an electric motor. A potentiometer is used to measure the cart's position through its rotation, while another potentiometer measures the angle of the pendulum.

If the pendulum's angle is taken as the output relative to the vertical axis (in its upright position), in that case, it becomes evident that the system is inherently unstable, as the pendulum will tip over if released at even a slight angle. To stabilize the system and maintain the pendulum in an upright position, a feedback control system was required.

The overall block diagram for the feedback control system of the inverted pendulum is shown below.

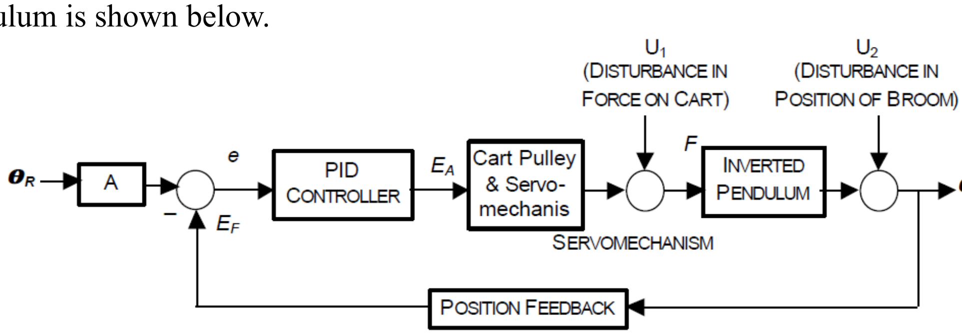


Fig. 2 Feedback control system for the IP

In our implementation, feedback is utilized only from the pendulum's angle, meaning only one of the four states is considered for feedback. The other states—cart position, cart velocity, and pendulum angular velocity—are not included. This setup could be improved by incorporating a control loop for the cart's position.

Initially, the pendulum is manually positioned upright in an unstable equilibrium or given a small initial displacement. Once this setup is in place, the controller is activated to balance the pendulum and maintain stability despite disturbances. Simple disturbances might include a gentle tap on the pendulum, while more complex disturbances could involve wind gusts generated by a fan.

This configuration serves as a platform to explore the control of an open-loop unstable system and demonstrates how feedback control can stabilize such systems. Various control strategies, from simple phase advance compensators to advanced neural network controllers, can be implemented and studied using this setup.

3.2 THE MATHEMATICAL MODEL

An inverted pendulum represents a fundamental problem in control systems. It is a nonlinear and unstable system with a single input signal and multiple output signals. The primary objective is to maintain the pendulum in its vertical position while mounted on a motor-driven cart.

The diagram below illustrates an inverted pendulum. The goal is to move the cart along the x-axis to a desired position without allowing the pendulum to tip over. A DC motor powers the cart, and in this implementation, it is controlled using an analog controller. The cart's position along the x-axis and the pendulum's angle$(theta - \theta)$ are measured and fed into the control system. Additionally, an external force (F) can be applied to the top of the pendulum to introduce disturbances.

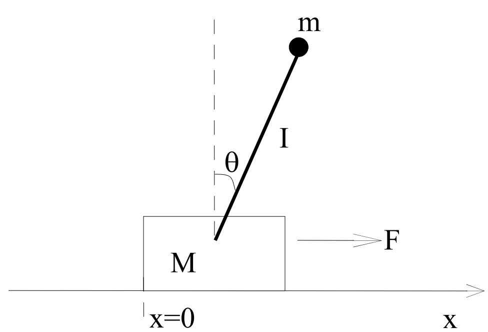


Fig. 3 Inverted pendulum schematic

3.2.1 Inverted Pendulum System Equations

First, mathematical representations must be used to comprehend the system's behavior. An illustration of the cart and pendulum with the relevant forces is shown in Figure 3. A list of the pertinent factors and their definitions is given below.

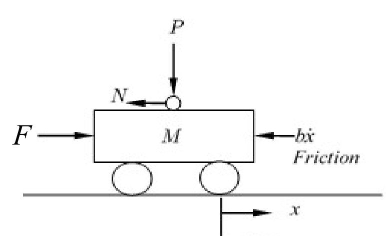


Fig. 4 Paramete

3.2.2 Derivation of the equations of motion

We will now derive the equations of motion using the Euler-Lagrange equations.

The Lagrangian as given by equation (1) becomes

$$L = K - V = \tfrac{1}{2}(M+m)\dot{x}^2 + \tfrac{1}{2}ml^2\dot{\theta}^2 - ml\dot{\theta}\dot{x}\cos\theta + \tfrac{1}{2}I\dot{\theta}^2 - (mgl\cos\theta) \quad (8)$$

3.2.3 The Euler-Lagrange Equations

With the explicit expression for the Lagrangian *L*, the Euler-Lagrange equations can be formulated. The general formula for the Euler-Lagrange is.

$$\frac{d}{dt}\left[\frac{\partial L(q,\dot{q})}{\partial \dot{q}}\right] - \frac{\partial L(q,\dot{q})}{\partial q} = F_{ext} \quad (9)$$

where $(q,\dot{q})$ are the generalized coordinates which are the coordinates chosen within the system. For our system $q = x,\ \dot{q} = \dot{x}$ for horizontal motion. For the angular motion; however, $q = \theta,\ \dot{q} = \dot{\theta}$. F_{ext} Represents non-conservative external forces.
Thus, for the horizontal motion, our Euler-Lagrange equation reads.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = F \quad (10)$$

For the angular motion, the Euler-Lagrange equation reads.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad (11)$$

The generic version of the Euler-Lagrange equation is shown in equation (9). Equation (10) gives the precise Euler-Lagrange equation about the position of the cart, whereas equation (11) gives the details of the one referring to the angle of the pendulum.

$$\Rightarrow \qquad L = \tfrac{1}{2}(M+m)\dot{x}^2 + \tfrac{1}{2}ml^2\dot{\theta}^2 - ml\dot{\theta}\dot{x}\cos\theta + \tfrac{1}{2}I\dot{\theta}^2 - mgl\cos\theta$$

$$\frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} - ml\dot{\theta}\cos\theta \quad (12)$$

Furthermore,

$$\frac{\partial L}{\partial \theta} = ml\dot{x}\dot{\theta}\sin\theta + mgl\sin\theta, \quad \frac{\partial L}{\partial \dot{\theta}} = (ml^2 + I)\dot{\theta} - ml\dot{x}\cos\theta \quad (13)$$

Hence

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (M+m)\ddot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta \quad (14)$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = (ml^2 + I)\ddot{\theta} - ml\ddot{x}\cos\theta + ml\dot{\theta}\dot{x}\sin\theta \quad (15)$$

The equations of motion are then;

$$(M+m)\ddot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta = F - b\dot{x} \tag{16}$$

$$(ml^2 + I)\ddot{\theta} - ml\ddot{x}\cos\theta - mgl\sin\theta = 0 \tag{17}$$

3.2.4 Linearization of the system

Equations (16) and (17), determine the transfer functions for the pendulum's angle and the cart's location, which must first be linearized before the model is simulated in MathCAD 14 (2014). In this instance, it is considered that the typical pendulum starts in its vertical downward position and swings through tiny angles, we thus have the following approximations

$$\theta \to 0,\ \cos\theta \approx 1,\ \sin\theta \approx \theta,\ \dot{\theta} \approx 0 \tag{18}$$

Furthermore, we make another simplifying assumption that the moment of inertia I and frictional force $b\dot{x}$ are negligible. This results in the linearized equations of motion given by;

$$(M+m)\ddot{x} - ml\ddot{\theta} = F \tag{19}$$

$$ml^2\ddot{\theta} - ml\ddot{x} - mgl\theta = 0 \tag{20}$$

Dividing (20) by ml we get

$$l\ddot{\theta} = g\theta + \ddot{x} \tag{21}$$

Substituting (21) in (19) we get

$$(M+m)\ddot{x} - m(g\theta + \ddot{x}) = F$$

$$\Rightarrow \quad M\ddot{x} = F + mg\theta \tag{22}$$

Substituting (22) into (21), we get

$$l\ddot{\theta} = g\theta + \frac{1}{M}F + \frac{m}{M}g\theta = \left(1 + \frac{m}{M}\right)g\theta + \frac{1}{M}F$$

$$\Rightarrow \quad \ddot{\theta} = \frac{1}{lM}F + \frac{g}{lM}(M+m)\theta \tag{23}$$

Thus equations (19) and (20) are simplified to

$$\ddot{x} = \frac{1}{M}F + \frac{m}{M}g\theta \tag{24}$$

$$\ddot{\theta} = \frac{1}{lM}F + \frac{g}{lM}(M+m)\theta \tag{25}$$

3.2.5 State-space

A different method for expressing the input and output behavior of the system than the transfer function is state-space. The position of the cart and the angle of the pendulum can be combined into a single statement using the state-space model. The variables in the two linearized equations of motion (24 and 25) need to be re-expressed using additional variables to be converted into state-space form. Equation (26) provides the many variables that correspond to these in matrix form.). It is thus possible to develop the equations of motion for the variables specified in (26). We start by determining the variables.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{pmatrix} \tag{26}$$

$$\dot{x}_1 = x_2 \tag{27}$$

$$\dot{x}_3 = x_4 \tag{28}$$

$$\dot{x}_2 = \frac{1}{M}F + \frac{m}{M}gx_3 \tag{29}$$

$$\dot{x}_4 = \frac{1}{lM}F + \frac{g}{lM}(M+m)x_3 \tag{30}$$

In matrix form, we have;

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{m}{M}g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{lM}(M+m) & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{lM} \end{pmatrix} F \tag{31}$$

The above can now be written more concisely as;

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu \tag{32}$$

$u = F$

This state-space representation can be simulated in MathCAD 14.

3.3 Mathcad Simulation of the Linearized Inverted Pendulum Dynamics

We now consider the simulation of the equations (27) to (30) using the computational algebra software MathCAD 14. The stabilizing force F is represented as $F = a\cos(\omega t)$ where a is the amplitude. $M=5, m=2, g_0=9.8, I=0.9, w=10, a=10$

Define a function that determines a vector of derivative values of any point (t, x)

$$D(t,X) = \begin{bmatrix} x_1 \\ \frac{1}{M}acos\,(\omega t) + \frac{m}{M} \quad g_0 x_2 \\ x_3 \\ \frac{1}{M}acos\,(\omega t) + \frac{g_0}{M}(M+m)x_2 \end{bmatrix}$$

Define additional argument for the ODE solver

$t_0 = 0$ Initial value of independent variable
$t_1 = 2.5$ Final value of independent variable

$X_0 = \begin{bmatrix} 0 \\ 1 \\ 0.01 \\ 0 \end{bmatrix}$, Vector of initial function values

$N = 1 \times 10^3$ Number of solution values on $[t_0, t_1]$

$SI = Rkadapt(x_0, t_0, t_1, N, D)$ Runge-Kuta Numerical Algorithm

$t = SI^{\langle 0 \rangle}$ Independent Variable values

$x_1 = SI^{\langle 1 \rangle}$ First solution function values

$x_2 = SI^{\langle 2 \rangle}$ Second solution function values

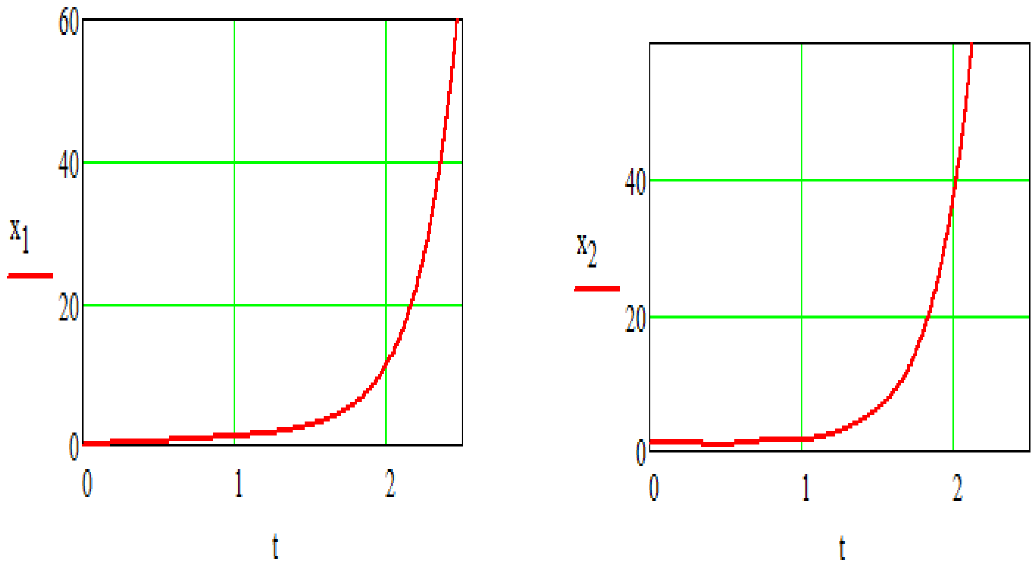
$x_3 = SI^{\langle 3 \rangle}$ Third solution function values

$x_4 = SI^{\langle 4 \rangle}$ Fourth Solution function values

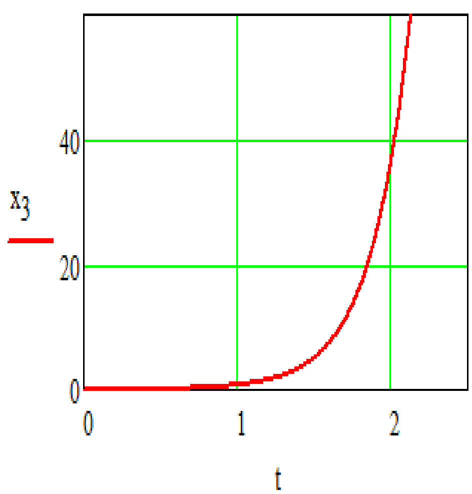
Table 1: Solution Matrix, SI

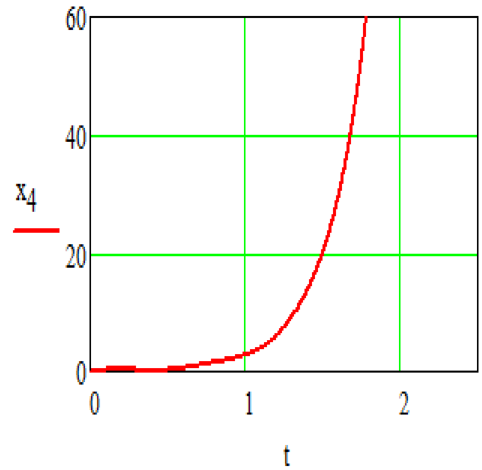
	0	1	2	3	4
0	0	0	1	0.01	0
1	2.5×10^{-3}	2.506×10^{-3}	1.005	0.01	5.025×10^{-3}
2	5×10^{-3}	5.025×10^{-3}	1.01	0.01	0.012
3	7.5×10^{-3}	7.557×10^{-3}	1.015	0.01	0.018
4	0.01	0.01	1.02	0.01	0.024
5	0.013	0.013	1.025	0.01	0.03
6	0.015	0.015	1.03	0.01	0.036
7	0.018	0.018	1.036	0.01	0.041
8	0.02	0.02	1.041	0.01	0.047
9	0.023	0.023	1.046	0.011	0.053
10	0.025	0.026	1.05	0.011	0.059
11	0.028	0.028	1.055	0.011	0.065
12	0.03	0.031	1.06	0.011	0.07
13	0.033	0.034	1.065	0.011	0.076
14	0.035	0.036	1.07	0.011	0.082
15	0.038	0.039	1.075	0.012	...

3.2.6 Graphical profiles of the solutions



(b)





(c) (d)

Fig. 14 Graphical profiles of the solutions

DISCUSSIONS

We discuss the major results from the paper. The inverted pendulum has an unstable equilibrium point at the upright position. The Pendulum dynamics are non-linear exhibiting chaotic behavior. This is depicted in Figure 14 -graphical profiles of the solutions (a, b, c, d). The graphs are all asymptotic around t = 2 minutes. Remembering that;

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{pmatrix}$$

Thus the distance x traveled by the mass *m* in the process of stabilization is unbounded, similarly, for the velocity $\dot{x}$, the angle θ changes repeatedly many times and so is also unbounded, and the same for the angular velocity $\dot{\theta}$. Furthermore, we assume the stabilizing force *F* is represented by $F = a\cos(\omega t)$, where *a,* is the amplitude. This explains the profile of the graphs.
Sensitivity to Initial Conditions: Small changes in initial conditions lead to significantly different outcomes.
Bifurcations: Changes in control parameters caused sudden shifts in pendulum behavior.

CONCLUSION

In conclusion, the work has been able to use the generalized Euler-Lagrange equation to derive the equations of motion, and a powerful computational algebra software called MathCAD 14 was used in this research to provide the solution efficiently. The graphical profiles of the solutions (a, b, c, d) are all asymptotic around $t = 2$ minutes. Hence we were able to successfully stabilize the pendulum, enabling precise and rapid control of the carriage's position on the track.

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