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NON-HYBRID CONTINUOUS MULTISTEP METHOD FOR BLOCK EXTENDED SECOND DERIVATIVE BACKWARD DIFFERENTIATION FORMULA

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ABSTRACT

This research paper, the solution of stiff problems by non- hybrid continuous multistep method for block extended second derivative backward differentiation formula is presented. The scheme was constructed through interpolation and collocation concept. The derived method is tested and region of absolute stability is also verified. The approximate solution indicates that to be efficient with better accuracy when compared with other author's results.

1. Introduction

Ordinary differentials are mathematical model of physical phenomena in science and engineering that contain derivatives of an unknown function of one or more several variable always leads to initial value problem of the form $y' = f(x, y)$ with $y(\alpha) = \eta$ (1)

In whatever manner, few of restricted sequence of numerical schemes are proposed for solution of stiff ordinary differential equations with set of additional constraints. Many authors have developed method to approximate solution to (1) relating to numbers by reducing it to system of first order stiff equations. The block schemes produce approximate results with less than computational work as to equate to non-block method [1].

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The modification and extension show some superiority over the conventional Backward Differentiation formulae method in terms of accuracy and computational cost [2]. The construction of continuous scheme had been overpowered in increasing concern direct to the fact that it adored certain benefit [3]. Introduced a direct integration implicit variable **steps method for solving higher order systems of ordinary differential** equations (ODEs). [4] Have solved many stiff problems in (1) presently with hybrid at the function of continuous scheme and the discrete method at more than one grid point simultaneously. According [5] and [6] block method was firstly proposed by [7] who advocated the use of block as a means of getting a starting value for predictor-corrector algorithm and later adopted as a full method. Their work motivated us to develop the non-hybrid four step continuous multistep method for block extended second derivative backward differentiation formula.

Derivation of the Method

We consider a power series as a basic function for approximation of:

$$y(x)=\sum_{j=0}^{p+q-1} a_j x^j \tag{2}$$

Where p and q are number of distinct interpolation and collocation respectively.

$$f(x, y, y')=\sum_{j=0}^{p+q-1} j(j-1)a_j x^{j-2} \tag{3}$$

Differentiating equation (2) two times and substitute into equation (1) to give,

Now interpolating (2) at point x_{n+p}, $p=0,1,2$ *and* 3 and collocating (3) at points x_{n+q}, $q=3$*and* 4, at lead to a system of equation written below

$$AX=Y \tag{4}$$

Where,

$$\begin{bmatrix} x_n^0 & x_n^1 & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ x_{n+1}^0 & x_{n+1}^1 & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 \\ x_{n+2}^0 & x_{n+2}^1 & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 \\ x_{n+3}^0 & x_{n+3}^1 & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 \\ 0 & x_{n+3}^0 & 2x_{n+3}^1 & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 & 6x_{n+3}^5 & 7x_{n+3}^6 \\ 0 & x_{n+4}^0 & 2x_{n+4}^1 & 3x_{n+4}^2 & 4x_{n+4}^3 & 5x_{n+4}^4 & 6x_{n+4}^5 & 7x_{n+4}^6 \\ 0 & 0 & 2 & 6x_{n+3}^1 & 12x_{n+3}^2 & 15x_{n+3}^3 & 30x_{n+3}^4 & 42x_{n+3}^5 \\ 0 & 0 & 2 & 6x_{n+4}^1 & 12x_{n+4}^2 & 15x_{n+4}^3 & 30x_{n+4}^4 & 42x_{n+4}^5 \end{bmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ f_{n+3} \\ f_{n+4} \\ g_{n+3} \\ g_{n+4} \end{pmatrix}$$

Using Gaussian elimination method on (4) to solved for the a_j's. The values of the a_j's obtained and after some manipulations, this gives a non-hybrid continuous linear multistep method of the form;

$$y(x)=\alpha_0(x)y_n+\alpha_1(x)y_{n+1}+\alpha_2(x)y_{n+2}+\alpha_3(x)y_{n+3}+h[\sum_{j=3}^{4}\beta_j(x)f_{n+j}]+h^2\left[\sum_{j=3}^{4}\gamma_v(x)g_{n+v}\right],\ j=3,4v_i=3,4$$

(5)

Where the continuous coefficient (5) of the method are given as:

$$\alpha_0(x)=1-\frac{4302x}{1447h}+\frac{31013x^2}{8682h^2}-\frac{712991x^3}{312552h^3}+\frac{131867x^4}{156276h^4}-\frac{1579x^5}{8682h^5}+\frac{3319x^6}{156276h^6}-\frac{325x^7}{312552h^7}$$

$$\alpha_1(x)=\frac{16740x}{1447h}-\frac{33021x^2}{1447h^2}+\frac{107019x^3}{5788h^3}-\frac{91279x^4}{11576h^4}+\frac{21627x^5}{11576h^5}-\frac{2701x^6}{11576h^6}+\frac{139x^7}{11576h^7}$$

$$\alpha_2(x)=-\frac{69714x}{1447h}+\frac{344601x^2}{2894h^2}-\frac{1300707x^3}{11576h^3}+\frac{309731x^4}{5788h^4}-\frac{39891x^5}{2894h^5}+\frac{10635x^6}{5788h^6}-\frac{1153x^7}{11576h^7}$$

$$\alpha_3(x)=\frac{57276x}{1447h}-\frac{433345x^2}{4341h^2}+\frac{15026527x^3}{156276h^3}-\frac{14524675x^4}{312552h^4}+\frac{420127x^5}{34728h^5}-\frac{508001x^6}{312552h^6}+\frac{27703x^7}{312552h^7}$$

$$\beta_3(x)=-\frac{37784x}{1447}+\frac{94846x^2}{1447h}-\frac{1628897x^3}{26046h^2}+\frac{1547693x^4}{52092h^3}-\frac{43601x^5}{5788h^4}+\frac{50947x^6}{52092h^5}-\frac{2669x^7}{52092h^6}$$

$$\beta_4(x)=-\frac{9909x}{1447}+\frac{26919x^2}{1447h}-\frac{113949x^3}{5788h^2}+\frac{30569x^4}{2894h^3}-\frac{4428x^5}{1447h^4}+\frac{1321x^6}{2894h^5}-\frac{159x^7}{5788h^6}$$

$$\gamma_3(x)=\frac{23496th}{1447}-\frac{61882x^2}{1447}+\frac{377521x^3}{8682h}-\frac{386641x^4}{17364h^2}+\frac{35473x^5}{5788h^3}-\frac{15047x^6}{17364h^4}+\frac{859x^7}{17364h^5}$$

$$\gamma_4(x)=\frac{2808th}{1447}-\frac{61882x^2}{2894}+\frac{32823x^3}{5788h}-\frac{8911x^4}{2829h^2}+\frac{1310x^5}{1447h^3}-\frac{199x^6}{1447h^4}+\frac{49x^7}{5788h^5}$$

On evaluating (5) at all point $x_n, x_{n+1}, x_{n+2}\ and\ x_{n+4}$ yields the following discrete methods are obtained

$$y_n-\frac{198126}{31031}y_{n+1}+\frac{1033803}{31031}y_{n+2}-\frac{866690}{31031}y_{n+3}=\frac{6h}{31031}[94846f_{n+3}+26919f_{n+4}]+\frac{3h^2}{31031}[144g_n+123764g_{n+3}+15363g_{n+4}]$$

$$y_n+\frac{13419}{4678}y_{n+1}-\frac{80811}{2339}y_{n+2}+\frac{143525}{4678}y_{n+3}=\frac{3h}{2339}[16039f_{n+3}+4086f_{n+4}]+\frac{3h^2}{2339}[1447g_n-9870g_{n+3}-1150g_{n+4}]$$

$$y_n - \frac{7038}{259}y_{n+1} + \frac{9171}{259}y_{n+2} - \frac{15950}{259}y_{n+3} = \frac{6h}{259}[2062f_{n+3} + 783f_{n+4}] - \frac{36}{259}[1447g_n + 2038g_{n+3} + 214g_{n+4}]$$

$$y_{n+4} + \frac{1}{1447}y_n - \frac{16}{1447}y_{n+1} + \frac{216}{1447}y_{n+2} - \frac{1648}{1447}y_{n+3} = \frac{84h}{1447}[8f_{n+3} + 7f_{n+4}] + \frac{3h^2}{2339}[4g_n - g_{n+4}]$$

ANALYSIS OF THE BASIC PROPERTIES OF THE METHOD

Consider the linear operator $L\{y(x):h\}$ defined by

$$L\{y(x):h\} = A^{(0)}Y_m^{(i)} - \sum_{i=0}^{k}\frac{jh^{(i)}}{i!}y_n^{(i)} - h^{(3-1)}\left[d_i f(y_n) + b_i F(Y_m)\right] \tag{6}$$

Using the Taylor series expansion of Y_m *and* $F(Y_m)$ and comparing the coefficients of h gives

$$L\{y(x):h\} = C_0 y(x) + C_1 y'(x) + \cdots + C_p h^p y^p(x) + C_{p+1}h^{p+1}y^{p+1}(x) + C_{p+2}h^{p+2}y^{p+2}(x) + \cdots \tag{7}$$

The linear operator L and the associate block method are said to be of order p if $C_0 = C_1 = \cdots = C_p = C_{p+1} = 0 \quad C_{p+2} \neq 0.$ C_{p+2} is called the error constant and implies that the truncation error is given by $t_{n+k} = C_{p+2}h^{p+2}y^{p+2}(x) + 0h^{p+3}$

Using the concept above, the non-hybrid block methods is constructed using MAPLE 18 SOFTWARE gives the following uniform order and error constants.

Comparing the coefficient of h, according to [8].

Table 1 Order and Error constant of the Method

$y(x_{n+j})$	**Order**	**Error Constant**
$j = 0$	7	$-\frac{171639}{8683640}$
$j = 1$	7	$\frac{21111}{1309840}$
$j = 2$	7	$\frac{1217}{31080}$
$j = 3$	7	$-\frac{3}{50645}$

REGION OF ABSOLUTE STABILITY OF THE METHOD

The hybrid block method is said to be absolutely stable, if for a given h, all roots of the characteristic polynomial $\pi(z,h) = \rho(z) - \bar{h}\sigma(z)$, satisfied $|z_t| < 1$.

Applying the boundary locus method, after some manipulation, then substituting the stability polynomial and obtain the region of absolute stability [9]

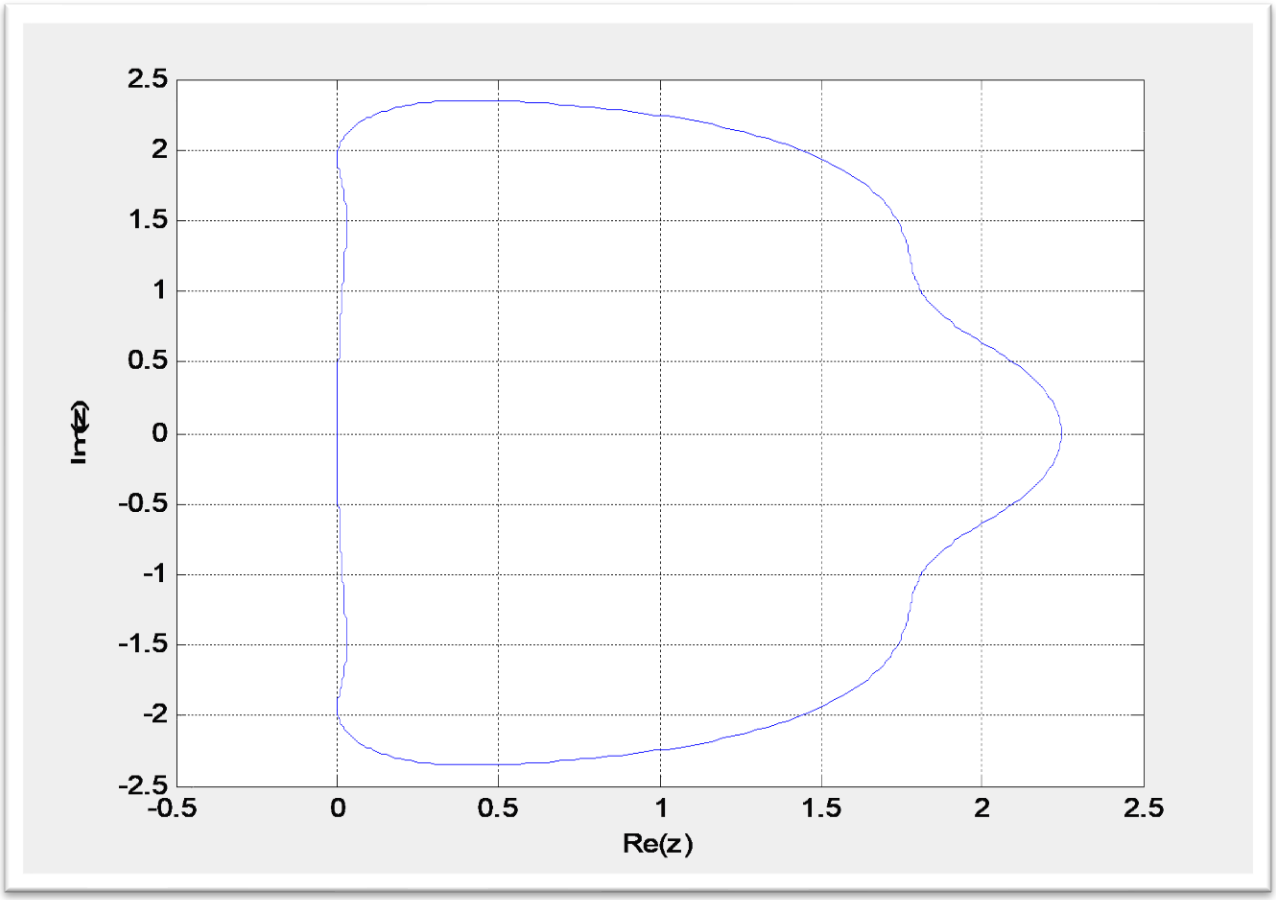

Figure 1. Absolute Stability Region of Non-Hybrid block four step- method

NUMERICAL EXPERIMENT AND RESULTS

The newly constructed continuous non-hybrid second derivative block backward differentiation formula are applied in block form for step numbers k = 4 to solve three problem and results were compared with results from existing methods.

4SNHCMMBESDBDF: Four- Step Non-Hybrid Continuous Multistep Method for Block Extended Second Derivative Backward Differentiation Formula

Problem 1

$$y_1' = -29998y_1 - 59994y_2$$

$$y_2' = 9999y_1 + 19997y_2$$

Exact $y_1(x) = \left(\frac{1}{9999}\right)(29997e^{-10000x}\text{-}19998e^{-x})y_1(0) = 1$

$y_2(x) =$ $\text{-}e^{-10000x}{+}e^{-x}y_2(0) = 0$

$with\ h = 0.01$

Table 2: Absolute errors of numerical solutions of problem 1 solve with the method$k = 4$

x	4SNHCMMBESDBDF $f(y_1)$	4SNHCMMBESDBDF $f(y_2)$
0.1	5.51156E-10	2.75566E-10
0.2	7.26110E-11	3.63010E-11
0.3	5.17330E-11	2.58680.E-11
0.4	5.11580E-11	2.55800E-11
0.5	2.77330.E-11	1.38670E-11
0.6	1.45410E-11	7.27000E-12
0.7	626800E-12	3.13400E-12
0.8	2.80800E-12	1.40400E-12
0.9	1.20500E-12	6.02000E-13
1.0	4.99000E-13	2.50000E-13

Problem 2

$$y_1' = -2y_1 + y_2 + 2sinx$$

$$y_2' = 998y_1 - 999y_2 + 999(cosx - sinx) \qquad with\ h = 0.01$$

Exact $\quad y_{1(x)} = 2e^{-x} + sinx y_1(0) = 2$

$$y_2(x) = 2e^{-x} + cosx y_2(0) = 3$$

Table 3: Absolute errors of numerical solutions of problem 2 solve with the method$k = 4$

x	4SNHCMMBESDBDF $f(y_1)$	4SNHCMMBESDBDF $f(y_2)$
0.1	4.73966E-03	3.21930E-03
0.2	1.61452E-02	1.66972E-02
0.3	3.01153E-02	2.91914E-02

0.4	1.93145E-02	1.77641E-02
0.5	8.17097E-03	8.2233E-03
0.6	2.77493E-02	2.70109E-02
0.7	2.16698E-02	2.01205E-02
0.8	4.38621E-02	5.32198E-02
0.9	2.64292E-02	2.58911E-02
1.0	2.41806E-02	2.26633E-02

Problem 3

$$y_1' = 0.01y_1 - y_2 + y_3$$

$$y_2' = y_1 - 100.005y_2 + 99.995y_3$$

$$y_3' = 2y_1 + 99.995y_2 - 100.005y_3$$

Exact $y_{1(x)} = e^{-0.01x}(cos2x + sin2x)\ y_1(0) = 1$

$y_2(x) = e^{-0.01x}(cos2x + sin2x)$ $+e^{-200x}$ $y_2(0) = 1$

$y_2(x) = e^{-0.01x}(cos2x + sin2x)$ $-e^{-200x}$ $y_3(0) = 1$

$with\ h = 0.1$

Table 4: Absolute errors of numerical solutions of problem 3 solve with the method k =4

x	4SNHCMMBESDBDF (y_1)	4SNHCMMBESDBDF (y_2)	4SNHCMMBESDBDF (y_3)
0.1	2.00200E-12	5.92751E-03	0.00000E-00
0.2	5.45000E-13	2.37626E-04	0.00000E+00
0.3	1.11000E-10	3.81694E-06	0.00000E+00
0.4	2.00000E-11	1.35361E-08	0.00000E+00

0.5	3.00000E-15	9.497435-10	0.00000E+00
0.6	1.00000E-15	2.77430E-11	0.00000E+00
0.7	0.00000E+00	3.59000E-13	0.00000E+00
0.8	0.00000E+00	1.00000E-15	0.00000E+00
0.9	0.00000E+00	0.00000E+00	0.00000E+00
1.0	0.00000E+00	0.00000E+00	0.00000E+00

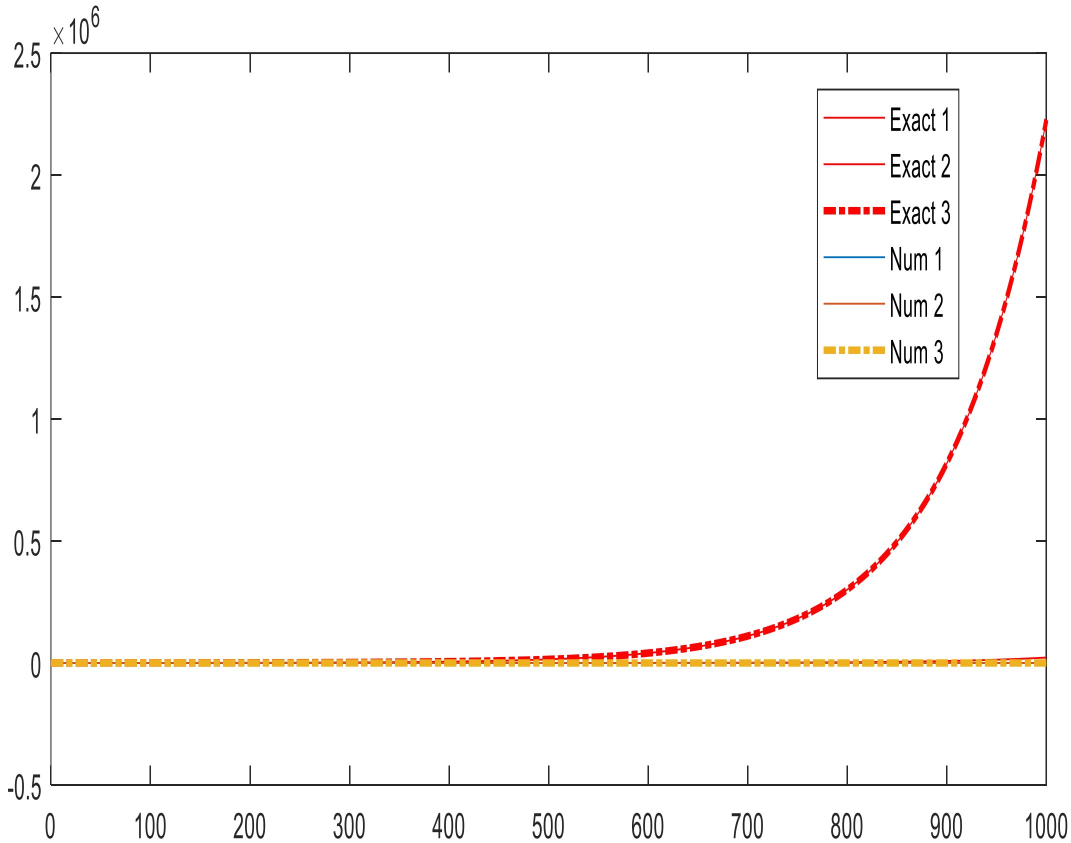

Figure 2: Curve Solution of Problem 3 Solved with 4SNHCMMBESDBDF

Table 5: Comparison of 4SNHCMMBESDBDF for problem 1

x	$Error\ in\ New\ Method$ (y_1)	$Error\ in\ New\ Method$ (y_2)	$Error\ in$ [10] (y_1)	$Error\ in$ [10] (y_2)
0.1	5.51156E-10	2.75566E-7	3.61E-07	3.60E-07
0.2	7.26110E-11	3.63010E-11	3.21E-07	3.30E-07
0.3	5.17330E-11	2.58680.E-11	6.28E-07	3.27E-07
0.4	5.11580E-11	2.55800E-11	5.65E-07	5.65E-07
0.5	2.77330.E-11	1.38670E-11	6.69E-07	6.68E-07
0.6	1.45410E-11	7.27000E-12	6.03E-07	6.02E-07
0.7	626800E-12	3.13400E-12	5.92E-07	5.92E-07
0.8	2.80800E-12	1.40400E-12	5.36E-07	5.37E-07
0.9	1.20500E-12	6.02000E-13	7.38E-07	7.38E-07
1.0	4.99000E-13	2.50000E-13	6.70E-07	6.70E-07

CONCLUSION

In this paper, we have proposed a new block numerical scheme for the solution of first-order stiff ordinary differential equations. The method was applied to solve three stiff problems. Maple and Matlab were used to generate the scheme and numerical solutions. Three numerical problem have been used to test the accuracy and efficiency of the developed method. The region of absolute stability analysis of newly developed method to be zero stable, consistent and convergent. Finally, we conclude that our method gave better accuracy than existing method compared in Table 5 and numerical Curve solutions compared with exact solutions is successful.

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