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ON THE CONSTRUCTION AND EVALUATION OF NINE-TREATMENT INCOMPLETE-BLOCK DESIGNS OF ORDER THREE FROM SOME QUASI-SEMI-LATIN SQUARES

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ABSTRACT

The efficiency and optimality characterization of eight incomplete-block designs from some quasi-semi-Latin squares of order three were evaluated in this study. The $(3 \times 3)/3$ quasi-semi-Latin squares were constructed from three distinct and unique $(3 \times 3)/2$ quasi-semi-Latin squares. Incomplete-block designs were generated from the $(3 \times 3)/3$ quasi-semi-Latin squares by considering different block structures: 'short' rows, 'little' columns and alternate treatment positions. The variance of the treatment contrasts, $\text{var}(\hat{\tau}_i - \hat{\tau}_j)$, $i \neq j$, show that design Λ_{12} is near-variance-balanced. Treatment pairs estimated with the same variance have the same efficiency while treatment pairs with minimum variances are the most efficient. The designs, Λ_{11}, Λ_{21} and Λ_{31}, constructed using "little" columns as blocks have the same A-, D-, E- and MV-optimality criteria. The IBDs constructed using "little" columns as blocks led to about 8 % loss of information while the designs constructed using "short" rows as blocks led to about 39 % loss of information.

1. Introduction

Let $\Lambda = \Lambda(v, b, r, k)$ be a v −treatment binary incomplete-block design arranged in b blocks of size k where each treatment is replicated r times and r is a constant. Each treatment pair in D appears ∂_{ij} times within blocks, $i = 1, 2, ..., v; j = 1, 2, ..., b$. [8] Provided some useful examples of applications of incomplete-block designs and details of their intra- and inter-block analysis. Other useful literature for the uses, construction and analysis of incomplete-block designs are [7], [11] and [12].

[6] Constructed and assessed the optimality and efficiency properties of six-treatment incomplete-block designs from some quasi-semi-Latin square. The quasi-semi-Latin squares were developed by [4] as three-factor block-structured combinatorial objects whose treatment

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entries are arranged as in the semi-Latin squares formation. Please see [1], [2] and [3] for further details on the structure of the semi-Latin squares. The arrangement of the semi-Latin square design has no regard for the block structure which is the point of deviation of the quasi-semi-Latin squares because the peculiar block arrangements are considered, see [5] and this bothers on the use of the "short" rows and "little" columns as blocks. The three-factor block-structured combinatorial designs whose entries have the formation of the semi-Latin square but with attention to the block structures were called quasi-semi-Latin squares by [5].

Three types of the three-factor block-structured quasi-semi-Latin squares were developed in [4] with their unique block characteristics. The first type of this QSLS design has m symbol (treatment) positions crossed with n rows and n columns for $n = 3$ and $m = 2$ (there are six treatments) and denoted by $(n \times m) \times n$. The second type has n rows crossed with n "big" columns where m "little" columns are nested in the n "big" columns denoted by $n \times (n \to m)$. The third type has n rows crossed with nw "little" columns and denoted by $(n \times nw)$. [6] explored the optimality and efficiency properties of incomplete-block designs developed by considering as blocks, the "short" rows and "little" columns of these types of quasi-semi-Latin squares.

There is yet no quasi-semi-Latin squares for $v > 6$ number of treatments. Therefore, in this paper, we consider the construction of nine-treatment ($v = 9$) quasi-semi-Latin squares and their corresponding incomplete-block designs. The statistical properties of the incomplete-block designs will be evaluated to ascertain the most efficient and optimal incomplete-block designs emanating from the nine-treatment quasi-semi-Latin squares. Also, the most efficient block structure for each pair of treatments were analyzed and ascertained. The next section (Section 2) shows the construction procedures for the $v = 9$ treatment QSLS. Section 3 covers the construction of incomplete-block designs from the QSLS while Section 4 is on the efficiency and optimality properties of the incomplete-block designs.

2. Construction of the Nine-Treatment Quasi-semi-Latin Squares

The method of superposition was used in constructing the nine-treatment QSLS in n rows and n columns where $n = 3$ with m treatment positions to obtain $(3 \times 3)/3$ QSLS. The method of superposition adopted here requires superimposing Latin square designs of order 3 on each of the three types of QSLS (see Figures 2, 3 and 4) constructed by [4] . Each QSLS in Figures 2, 3 and 4 contains six treatments, $v = 1, 2, 3, 4, 5$ and 6. Only one transformation set, presented as Figure 1, exists for the Latin square of order 3 , with treatments, $v = 7, 8$ and 9, which was used for superimposition. In this way, a total of three $(3 \times 3)/3$ quasi-semi-Latin squares are obtained, each belonging to the specific type of QSLS from which it was constructed (see Figures 5, 6 and 7). These QSLS's have nine treatments which are arranged in three rows, three columns and three treatments per row-column intersection. The treatments used in this study are 1, 2, 3, 4, 5, 6, 7, 8 and 9.

7	8	9
8	9	7
9	7	8

Figure1: Latin Square of Order 3

2 1	4 3	5 6
6 3	5 1	2 4
5 4	6 2	1 3

Figure 2: $(3 \times 3)/2$ Quasi-semi-Latin Square Type 1

1	4	6	2	5	3
6	3	1	5	4	2
2	5	4	3	1	6

Figure 3: $(3 \times 3)/2$ Quasi-semi-Latin Square Type 2

1	2	3	4	5	6
3	6	1	5	2	4
4	5	2	6	1	3

Figure 4: $(3 \times 3)/2$ Quasi-semi-Latin Square Type 3

2 7 1	4 8 3	5 9 6
6 8 3	5 9 1	2 7 4
5 9 4	6 7 2	1 8 3

Figure 5: $(3 \times 3)/3$ Quasi-semi-Latin Square Type 1

7	1	4	6	8	2	5	3	9
8	6	3	1	9	5	4	2	7
9	2	5	4	7	3	1	6	8

Figure 6: $(3 \times 3)/3$ Quasi-semi-Latin Square Type 2

7	1	2	3	8	4	5	9	6
8	3	6	1	9	5	2	7	4
9	4	5	2	7	6	1	8	3

Figure 7: $(3 \times 3)/3$ Quasi-semi-Latin Square Type 3

3. Binary Incomplete-Block Designs

There are different methods of constructing incomplete-block designs and some useful examples of these methods could be found in [13], [14], [15] and [16]. In this study, three specific block structures of the three $(3 \times 3)/3$ QSLS were considered from which the incomplete-block designs were constructed. The first set of incomplete-block designs were constructed by considering as blocks, the "little" columns of the QSLS in Figures 2, 3 and 4. The incidence matrices of the first incomplete-block designs (IBD), $\Lambda_{11}, \Lambda_{21}$ and Λ_{31}, emanating from this block arrangement are displayed in equations (1), (2) and (3) and denoted by $I_{\Lambda_{11}}, I_{\Lambda_{21}}$ and $I_{\Lambda_{31}}$, respectively, for Types 1, 2 and 3. The first subscript of the IBDs identifies the type of QSLS from which the IBD was developed while the second subscript is the identity of the block structure that gave the IBD from that type of QSLS. For instance, Λ_{21} is constructed from Type 2 QSLS with "little" columns as blocks; Λ_{33} is from Type 3 QSLS with the third block structure where the set first treatments appearing in each "Big" column of each row is a block and the set of second treatment appearing in each "Big" column of each row is another block. These incomplete-block designs are made binary by identifying treatment positions in each block by "1" and non-treatment positions by "0". That is, anywhere treatment occurred in a block of the incomplete-block design, the treatment is replaced by "1" while "0" is used to identify positions in the block where treatments did not occur (see [6]). Columns represent the blocks while rows are associated with the treatments.

$$I_{\Lambda_{11}} = \begin{matrix} 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{matrix} \qquad (1)$$

$$I_{\Lambda_{21}} = \begin{matrix} 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{matrix} \qquad (2)$$

$$I_{\Lambda_{31}} = \begin{matrix} 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{matrix} \qquad (3)$$

The second set of IBDs, Λ_{12} and Λ_{22}, were constructed by considering "short" rows of the QSLS as blocks. The incidence matrices, $I_{\Lambda_{12}}$ and $I_{\Lambda_{22}}$, of the IBD from types 1 and 2 QSLS are presented in equations (4) and (5). Type 3 has no clearly defined short rows since the three "long" rows are crossed by the six "little" columns.

$$I_{\Lambda_{12}} = \begin{matrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{matrix} \qquad (4)$$

$$I_{\Lambda_{22}} = \begin{matrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{matrix} \qquad (5)$$

The third set of IBD, $\Lambda_{13}, \Lambda_{23}$ and Λ_{33}, from types 1, 2 and 3 QSLS, respectively, are constructed by considering the set of first treatments appearing in each "Big" column of each row as block and the set of second treatments appearing in each "Big" column of each row as block. The incidence matrices, $I_{\Lambda_{13}}, I_{\Lambda_{23}}$ and $I_{\Lambda_{33}}$, of these incomplete-block designs are presented in equations (6), (7) and (8), respectively, for types 1, 2 and 3 QSLS.

$$I_{\Lambda_{13}} = \begin{array}{ccccccccc} 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array} \tag{6}$$

$$I_{\Lambda_{23}} = \begin{array}{ccccccccc} 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{array} \tag{7}$$

$$I_{\Lambda_{33}} = \begin{array}{ccccccccc} 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array} \tag{8}$$

Considering the three distinct block structures utilized in the construction of the IBD, only 8 incomplete-block designs are feasible. The next section is devoted to exploring the efficiency of these block structures.

4. Efficiency and Optimality Characterizations

The incomplete-block designs which emanated from the three types of QSLS were assessed to explore their statistical properties based on established statistical modelling procedures for incomplete-block designs, specifically, their efficiency and optimality characteristics. As pointed in [6], the precision with which treatment effects are estimated for each incomplete-block design could differ from one block structure to the other and these efficiency criteria are useful in measuring these differences.

For an experiment having a layout of an incomplete-block design, the underlying response of interest could be captured in the following model,

$$y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij}; \; i = 1, 2, \ldots, v \text{ and } j = 1, \ldots, b \qquad (9)$$

where y_{ij} is the response of interest, μ is the overall mean, τ_i is the ith treatment effect, β_j is the block effect and ϵ_{ij} is the uncorrelated random error with zero mean and common variance, σ_ϵ^2 (see [9] and [6]). The estimates of the treatment effects are of importance in this work and will be discussed subsequently.

4.1 Simple Contrasts

We first consider the simple contrasts from intra-block analysis. Let , τ_i be the i^{th} treatment effect and $\hat{\tau}_i$ be the least squares estimator of τ_i. Also, let the estimates of the simple contrast be $(\hat{\tau}_i - \hat{\tau}_j)$, $i \neq j$. Then, the variance of the treatment contrast, $\text{var}(\hat{\tau}_i - \hat{\tau}_j)$, $i \neq j$, from intra-block analysis of connected designs is given by

$$\text{var}(\hat{\tau}_i - \hat{\tau}_j) = (\phi_{ii} + \phi_{jj} - 2\phi_{ij})\sigma_k^2, \qquad (10)$$

where σ_k^2 is the plot-stratum variance for an incomplete-block design with blocks of size, k. The value of σ_k^2 is usually assumed to be 1 by authors. The term, ϕ_{ij}, is the ij^{th} element of the generalized inverse, Σ, of the information matrix, $\mathcal{H}$. For all the incomplete-block design considered here, $v = b$. The variances of the treatment contrasts for the eight incomplete-block designs are presented in Tables 1 to 8, respectively.

Treatment pairs with the same variance are estimated with the same efficiency. The smaller the variance of the treatment contrast, the higher the efficiency of estimation of the treatment pair(s). In Tables 1, 2, 3 and 6, the treatment pairs (7, 8), (7, 9) and (8, 9) are estimated with the same efficiency since they have the same variance of 0.6666; the pairs are estimated with the highest efficiency. Table 4 shows variances of treatment contrasts that are equal except for three pairs which also have the same but lower variances. By the definition given by [6], design Λ_{12} is near variance balanced. Treatment pairs estimated with variance of 0.6666 are estimated with the highest precision while treatment pairs estimated with variance of 1.1252 where estimated with the lowest precision.

Table 1: Variance and Efficiency Characteristics of Design Λ_{11}

Treatment Pair	Variance of Treatment Contrast	Pairwise Efficiency Factors
(2, 5)	0.7834	0.8510
(2, 6)	0.9334	0.7142
(5, 6)	0.7834	0.8510
(7, 8)	0.6666	1.0001
(7, 9)	0.6666	1.0001
(8, 9)	0.6666	1.0001
(1, 3)	0.7834	0.8510
(1, 4)	0.9334	0.7142
(3, 4)	0.7834	0.8510
(4, 5)	0.9334	0.7142
(4, 6)	0.7834	0.8510
(1, 2)	0.7834	0.8510
(2, 3)	0.9334	0.7142
(1, 5)	0.9334	0.7142
(3, 6)	0.9334	0.7142

Table 2: Variance and Efficiency Characteristics of Design Λ_{21}

Treatment Pair	Variance of Treatment Contrast	Pairwise Efficiency Factors
(7, 8)	0.6666	1.0001
(7, 9)	0.6666	1.0001
(8, 9)	0.6666	1.0001
(1, 2)	0.9334	0.7142
(1, 6)	0.7834	0.8510
(2, 6)	0.7834	0.8510
(3, 4)	0.9334	0.7142
(3, 5)	0.7834	0.8510
(4, 5)	0.7834	0.8510
(1, 4)	0.7834	0.8510
(4, 6)	0.9334	0.7142
(2, 3)	0.7834	0.8510
(2, 5)	0.9334	0.7142
(1, 5)	0.9334	0.7142
(3, 6)	0.9334	0.7142

Table 3: Variance and Efficiency Characteristics of Design Λ_{31}

Treatment Pair	Variance of Treatment Contrast	Pairwise Efficiency Factors
(7, 8)	0.6666	1.0001
(7, 9)	0.6666	1.0001
(8, 9)	0.6666	1.0001
(1, 3)	0.7834	0.8510
(1, 4)	0.9334	0.7142
(3, 4)	0.7834	0.8510
(2, 5)	0.7834	0.8510
(2, 6)	0.9334	0.7142
(5, 6)	0.7834	0.8510
(1, 2)	0.7834	0.8510
(2, 3)	0.9334	0.7142
(4, 5)	0.9334	0.7142
(4, 6)	0.7834	0.8510
(1, 5)	0.9334	0.7142
(3, 6)	0.9334	0.7142

Table 4: Variance and Efficiency Characteristics of Design Λ_{12}

Treatment Pair	Variance of Treatment Contrast	Pairwise Efficiency Factors
(1, 2)	0.8519	0.7826
(1, 7)	0.8519	0.7826
(2, 7)	0.6666	1.0001
(3, 6)	0.8519	0.7826
(3, 8)	0.6666	1.0001
(6. 8)	0.8519	0.7826
(4, 5)	0.8519	0.7826
(4, 9)	0.8519	0.7826
(5, 9)	0.6666	1.0001
(3, 4)	0.8519	0.7826
(4, 8)	0.8519	0.7826
(1, 5)	0.8519	0.7826
(1, 9)	0.8519	0.7826

(2, 6)	0.8519	0.7826
(6, 7)	0.8519	0.7826
(5, 6)	0.8519	0.7826
(6, 9)	0.8519	0.7826
(2, 4)	0.8519	0.7826
(4, 7)	0.8519	0.7826
(1, 3)	0.8519	0.7826
(1, 8)	0.8519	0.7826

Furthermore, all the treatment pairs for designs Λ_{11}, Λ_{21} and Λ_{31} are estimated with any of the three distinct variances, 0.6666, 0.7834 and 0.9334. Also, most of the treatment pairs that occurred simultaneously in each of the incomplete-block designs, Λ_{11}, Λ_{21} and Λ_{31}, are estimated with the same efficiency. This indicates that the block structure that gave rise to these IBDs is efficient. The most efficient block structure for estimation of treatment contrasts is using "short" rows as blocks because the IBD, Λ_{12}, which has the lowest and highest efficiencies as 0.6666 and 0.8519, respectively, giving the lowest range of variances of treatment contrasts. However, the same cannot be said of design Λ_{22} from the same block structure and whose lowest and highest variances are 0.7898 to 1.1252, indicating the influence of the specific type of QSLS on the quality of the IBD constructed from it. The most stable IBD are constructed from the Type 1 QSLS.

Table 5: Variance and Efficiency Characteristics of Design Λ_{22}

Treatment Pair	Variance of Treatment Contrast	Pairwise Efficiency Factors
(1, 4)	0.7931	0.8406
(1, 7)	0.9452	0.7053
(4, 7)	0.7899	0.8440
(3, 6)	0.7998	0.8335
(3, 8)	0.8794	0.7581
(6, 8)	0.7898	0.8441
(2, 5)	1.1252	0.5925
(2, 9)	1.0165	0.6558
(5, 9)	0.7899	0.8440
(2, 6)	0.7998	0.8335
(2, 8)	0.8794	0.7581
(1, 5)	0.7931	0.8406
(1, 9)	0.9452	0.7053
(3, 4)	1.1252	0.5925
(3, 7)	1.0165	0.6558
(4, 5)	0.9696	0.6876
(2, 3)	0.9090	0.7334
(7, 8)	1.0539	0.6326
(7, 9)	0.9696	0.6876
(8, 9)	1.0539	0.6326

Table 6: Variance and Efficiency Characteristics of Design Λ_{13}

Treatment Pair	Variance of Treatment Contrast	Pairwise Efficiency Factors
(2, 4)	0.7834	0.8510
(2, 5)	0.7834	0.8510
(4, 5)	0.9334	0.7142
(7, 8)	0.6666	1.0001
(7, 9)	0.6666	1.0001

(8, 9)	0.6666	1.0001
(1, 3)	0.7834	0.8510
(1, 6)	0.7834	0.8510
(3, 6)	0.9334	0.7142
(2, 6)	0.9334	0.7142
(5, 6)	0.7834	0.8510
(1, 4)	0.9334	0.7142
(3, 4)	0.7834	0.8510
(1, 5)	0.9334	0.7142
(2, 3)	0.9334	0.7142

Table 7: Variance and Efficiency Characteristics of Design Λ_{23}

Treatment Pair	Variance of Treatment Contrast	Pairwise Efficiency Factors
(5, 6)	1.0456	0.6376
(5, 7)	0.8104	0.8226
(6, 7)	0.8104	0.8226
(1, 3)	1.0456	0.6376
(1, 8)	0.8104	0.8226
(3, 8)	0.8104	0.8226
(2, 4)	1.0456	0.6376
(2, 9)	0.8104	0.8226
(4, 9)	0.8104	0.8226
(1, 4)	0.8104	0.8226
(1, 8)	0.8104	0.8226
(4, 8)	1.0456	0.6376
(2, 6)	0.8104	0.8226
(6, 9)	1.0456	0.6376
(3, 5)	0.8104	0.8226
(3, 7)	1.0456	0.6376
(1, 9)	1.0456	0.6376
(2, 7)	1.0456	0.6376
(5, 8)	1.0456	0.6376

Table 8: Variance and Efficiency Characteristics of Design Λ_{32}

Treatment Pair	Variance of Treatment Contrast	Pairwise Efficiency Factors
(3, 5)	0.7931	0.8406
(3, 7)	0.7899	0.844
(5, 7)	0.9452	0.7053
(1, 8)	0.7898	0.8441
(1, 9)	0.7998	0.8335
(8, 9)	0.8794	0.7581
(2, 4)	1.0165	0.6558
(2, 6)	1.1252	0.5925
(4, 6)	0.7899	0.844
(1, 2)	0.7998	0.8335
(2, 8)	0.8794	0.7581
(3, 9)	1.1252	0.5925
(7, 9)	1.0165	0.6558
(4, 5)	0.9452	0.7053
(5, 6)	0.7931	0.8406
(2, 9)	0.909	0.7334
(4, 7)	0.9696	0.6876

(4, 8)	1.0539	0.6326
(7, 8)	1.0539	0.6326
(3, 6)	0.9696	0.6876

4.2 Pairwise Efficiency Factors

According to [9] and [10], the pairwise efficiency factors of incomplete-block designs are useful in judging the usefulness and suitability of the designs. The pairwise efficiency factor for comparing each treatment pair of the $\Lambda = \Lambda(v, b, r, k)$ incomplete-block design is given by

$$E_{ij} = \frac{2}{rv_{ij}}, \quad (11)$$

where $v_{ij} = \phi_{ii} + \phi_{jj} - 2\phi_{ij}$, from equation (10). The higher the pairwise efficiency factor, the better the estimation of the treatment contrast (see [8]). The values of the pairwise efficiency factors for the pairs of treatments are displayed in Tables 1 to 8. The treatment contrasts, $(\hat{\tau}_7 - \hat{\tau}_8)$, $(\hat{\tau}_7 - \hat{\tau}_9)$ and $(\hat{\tau}_8 - \hat{\tau}_9)$ have pairwise efficiency factors of 1.0001, which implies that these contrasts are estimated with the same efficiency compared to the completely randomized design (CRD) of the same size. For the designs in Tables 1, 2, 3, 5, 6 and 7, most of the contrasts have their pairwise efficiency factors are about 80 % or more. This indicates that should the designs reduce the error variance by about 20 %, most of the treatment contrasts will be estimated with higher precision than the corresponding CRD of the same size.

4.3 Canonical Efficiency Factors and Optimality Criteria

Let $\mathcal{H}_\Lambda$ be the information matrix of an incomplete-block design and$\mathcal{H}_\Lambda^*$, the normalized information matrix, the canonical efficiency factors are the non-zero eigenvalues, λ_i, I = 1, 2, . . ., *v-1*, of $\mathcal{H}_\Lambda^*$. According to [10], canonical efficiency factors give useful summary about the properties of an incomplete-block design. The information matrix of the incomplete-block design is given by $\mathcal{H}_\Lambda = rI - \frac{1}{k} I_\Lambda I'_\Lambda$ where I is an identity matrix of order v while $\mathcal{H}_\Lambda^* = \frac{1}{r}\mathcal{H}_\Lambda$.

Three popular optimality criteria for evaluation of incomplete-block designs which are used in this study are based on the canonical efficiency factors of the design. They are the *A*-, *D*- and *E*-optimality criteria. The *A*-optimality criterion maximizes the harmonic mean of the canonical efficiency factors and is given by $= (v-1)\left(\sum_{i=1}^{v-1} \frac{1}{\lambda_i}\right)^{-1}$. The *D*-optimality criterion is the geometric mean of the canonical efficiency factors and is given by $D = \left(\prod_{i=1}^{v-1} \lambda_i\right)^{\frac{1}{v-1}}$. The *E*=optimality criterion minimizes the canonical efficiency factors of the IBDs and is expressed as $E = min(\lambda_i)$. Another optimality criterion used in evaluating the incomplete-block designs do not depend on the canonical efficiency factors is called the *MV*-optimality criterion. This optimality criterion depends on the pairwise efficiency factor and is expressed as the minimum of the pairwise efficiency factors such that $MV = min(E_{ij})$. The results of the optimality criteria for the IBDs are presented in Table 9.

Table 9: Design Optimality Criteria

Design	A-Optimality	D-Optimality	E-Optimality	MV-Optimality
Λ_{11}	0.916937	0.850741	0.5556	0.6666
Λ_{12}	0.615363	0.68658	0.3333	0.6666
Λ_{13}	0.916937	0.850741	0.5556	0.6666
Λ_{21}	0.916937	0.850741	0.5556	0.6666
Λ_{22}	0.61076	0.690065	0.2482	0.7898
Λ_{23}	0.586193	0.676842	0.2876	0.8104

Λ_{31}	0.916937	0.850741	0.5556	0.6666
Λ_{33}	0.61076	0.690065	0.2482	0.7898

The three designs, Λ_{11}, Λ_{21} and Λ_{31}, constructed by using "little" columns as blocks have the same *A*-, *D*-, *E*- and *MV*-optimality criteria. Another design that has the same *A*-, *D*-, *E*- and *MV*-optimality values like the other three is Λ_{13} which has a different block structure from the other three designs. Judging by their *A*-optimality, the designs constructed by using "little" columns as blocks lead to about only 8 % loss of information while the designs constructed by using "short" rows as blocks lead to about 39 % loss of information. The designs, Λ_{11}, Λ_{21} and Λ_{31}, have the best *A*-, *D*- and *E*-optimality values but displayed the worst *MV*-optimality values.

5. Conclusion

The results of the evaluation of the eight incomplete-block designs developed from three types of nine-treatment quasi-semi-Latin squares constructed by superposition were analyzed. The use of "little" columns as blocks gave incomplete-block designs that displayed the most stable variances of the simple contrast and the contrasts were estimated with the same efficiency as a completely randomized design of equal size. These incomplete block designs also have the best *A*-, *D*- and *MV*-optimality criteria compared to the other designs from the other block structures.

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