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MODELLING NONLINEAR DYNAMICAL SYSTEMS USING ITÔ STOCHASTIC DIFFERENTIAL EQUATIONS WITH JUMP PROCESS

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ABSTRACT

Diffusion processes have been used for describing nonlinear dynamical systems and the geometric Brownian motion has long served as a foundational model for capturing stochastic nature of processes characterized by the continuous random fluctuations. This study developed a modification to Lévy process for the Nigeria exchange rate to the US dollar using Ito stochastic differential equation by considering the case where the price movement involves sudden jumps, while capturing statistical features present in the time series. The developed extended xMJNID model, assumes that the market model has no arbitrage opportunities and the exchange rate follows a Merton model. The extension was a jump composed of Poisson process with nonconstant intensity function. Through simulation study and application to real data, the xMJNID model was shown to perform better than existing diffusion models including the Merton model and ARIMA. Comparisons were made using accuracy measures and Akaike and Bayesian information criteria.

1. Introduction

Recently there is a growing interest in the use of diffusion for describing nonlinear dynamics system such as financial indices since the variances changes through time. In particular, security prices in itself is a random process because of the actions of many different factors, both human and materials, which give rise to uncertainties in the system. Diffusion processes are important in several areas of science for modelling real life phenomena and can be characterized in terms of its infinitesimal generator [1]. The geometric Brownian motion (GBM) has long served as a foundational model for capturing stochastic nature of processes characterized by the continuous random fluctuations.

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In particular, this model has been argued to be well suited for forecasting diffusion processes [2], population dynamics [3] or most notably stock prices [4]. However, central to GBM is the assumption that logarithm of its solutions result in a normal distribution, which in real life may not fully encapsulate the complexities inherent in realistic systems. For instance, real data distributions often exhibit characteristics such as non-zero skewness, excess kurtosis or fluctuating volatility, notably deviating from the idealized bell curve [5. These deviations have significant implications for the accuracy of traditional GBM, by challenging its abilities to properly portray extreme events or interpret underlying dynamics of considered data [6].

Not many recent researches have tried to extend and improve the classical geometric Brownian motion model in various directions. Some authors try to provide a more realistic stochastic process for the underlying process by introducing a stochastic process for the volatility, but most of the application in the far past have focused on stock prices. One of such works is [7] who replaced Brownian motions by fractional Brownian motions in the diffusion model and added a compound Poisson jump and assuming that the variance of the stock return follows a fractional stochastic process. [8] presented a brief historical review of diffusions in Finance, followed by an even briefer discussion of jump-diffusions that involve Poisson or Lévy jumps. [9] considered the problem of computing the moment-generating function of the first exit time from the interval (a, b) for a time-homogeneous jump-diffusion process. Also on modelling stock prices, [10] investigated the fractional Brownian motion model with jump using Taylor expansion to deal with the control items in the model. [11] examined an optimal stopping problem for a GBM with random jumps. [12] gave a presentation of Black-Scholes-Merton logistic Brownian motion differential equation with jump diffusion. In order to deal with the sudden change of stock price and simplify the empirical research, [13] modified the jump-diffusion process factor of the fractional Brownian motion jump-diffusion process. [14] presented the logistic Brownian motion with jumps, but just like [15], the paper was also without practicality with real or simulated datasets. [16] looked at incorporating jumps and mean - reversion to forecast on logistic Brownian motion that could be used to predict prices of energy commodities.

The assumption that exchange rate is continuously modelled, is the basis for the use of Brownian motion. While this provides an interesting analysis used in the literature, it is however rather unrealistic. In real life, it has been observed that the dynamics of exchange rates contains discontinuities, hence the need for study of an extension that can cover the discontinuities. This study sought to develop a modification to Levy process for the exchange rate of Naira/US dollar, using Ito stochastic differential equation (SDE) by considering the case where the price movement involves sudden jumps, in the presence of seasonality, mean reversion and dependencies among exchange rates. Here the market model has no arbitrage opportunities and the exchange rate follows a Merton model [17]. An added innovation within this research is the inclusion of jumps composed of Poisson processes with nonconstant intensity function.

2 Methods

2.1 The Lévy Processes

Before we proceed in our proposed stochastic model for exchange rate of naira/dollar, we want to consider the Lévy processes. Lévy processes constitute an important family of stochastic processes, which includes Brownian motion as the only one that is continuous.

Definition of Lévy Process: [18]

A Lévy process $L(t)$ is a stochastic process on $(\Omega, \mathcal{F}, P)$ with the following properties:

1. $L(0) = 0$, P –a.s.
2. $L(t)$ has independent increments, that is, for $t_0 \leq t_1 \leq t_2 \leq \cdots$ we have that the random variables $L(t_0), L(t_1) - L(t_0), L(t_2) - L(t_1), \cdots$ are independent.
3. $L(t)$ has stationary increments, i.e., for all $s < t$ we have that $L(t) - L(s)$ has the same distribution as $L(t - s)$.
4. $L(t)$ is stochastically continues, i.e., for all $\epsilon < 0$, $\lim_{h \to 0} P(|L(t + h) - L(t)| \geq \epsilon) = 0$.
5. $L(t)$ has càdlàg paths, i.e., the trajectories are right-continuous with left limits.

Unlike with Brownian motion, there is no property of normal increments with the Lévy process. Property 4 implies that at any time t, the probability of a jump equals zero, i.e. we cannot have jumps at given times. Another example of Lévy process is the Poisson process $P(t)$ given by

$$\mathcal{P}(P(t) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \qquad t \geq 0$$

where $\lambda > 0$ is the intensity of the process. Moreover, a compound Poisson process $CP(t)$ is a process that sums a number of i.i.d. jumps sizes $Y(i)$ over a Poisson process $P(t)$,

$$CP(t) = \sum_{i=1}^{P(t)} Y(i), \qquad t \geq 0$$

where $\lambda > 0$ is the intensity and $CP(t)$ is independent of $Y(i)$. The compound Poisson process is e.g. widely used in property insurance to model the total claim amount in a portfolio, with the $Y(i)$'s representing the individual claim amounts and $CP(t)$ the number of claims in the portfolio.

Lévy measure [19]

Let $\left(L(t)\right)_{t \geq 0}$ be a Lévy process on $\mathbb{R}$. The measure v on $\mathbb{R}$ defined by

$$v(A) = E[\#t \in [0,1]: \Delta L_t \neq 0, \;\; \Delta L(t) \in A], \qquad A \in \mathfrak{B}(\mathbb{R})$$

is called the Lévy measure of L. That is, the Lévy measure denotes the expected number of jumps, per unit time, that belongs to A.

Next, we state the Itô-Lévy decomposition Theorem without proof, for which proof can be found in [19].

Itô-Lévy decomposition [20]

If $\left(L(t)\right)_{t \geq 0}$ is a Lévy process, then it has the decomposition

$$L(t) = \alpha t + \sigma W(t) + \int_{|z|<R} z\tilde{N}(t, dz) + \int_{|z|<R} zN(t, dz) \tag{2}$$

for some constants $\alpha, \sigma \in \mathbb{R}$ and $R \in [0, \infty]$. Moreover, $\tilde{N}(t, dz) = N(t, dz) - v(dz)dt$ is the compensated Poisson random measure of $L(t)$ and $W(t)$ is a Brownian motion which is independent of $N(t, dz)$.

The Itô-Lévy decomposition states that every Lévy process can be decomposed into a continuous Brownian motion with drift, a term incorporating the jumps that are smaller than some constant R and a term representing the jumps that are bigger or equal to R. The constant R can be chosen as small as we want, but since the case of infinitely many small jumps, i.e. $\int_{|z|<R} |z| dz = \infty$, could occur we need to compensate the Poisson random measure $N(dt, dz)$ around 0.

2.2 The Merton Jump Process Model

Let $X(t)$ be the asset price at time t, the log-return of $X(t)$, $\ln\left(\frac{X(t)}{X(0)}\right)$ is modelled as an exponential Lévy process $L(t)$ according to [17] such that

$$\ln\left(\frac{X(t)}{X(0)}\right) = L(t) = \left(\alpha - \frac{\sigma^2}{2} - \lambda\bar{k}\right)t + \sigma W(t) + \sum_{i=1}^{P(t)} K_i. \tag{3}$$

where $W(t)$ is a standard Brownian motion process, the term $\left(\alpha - \frac{\sigma^2}{2} - \lambda\bar{k}\right)t + \sigma W(t)$ is a Brownian motion with drift process and $\sum_{i=1}^{L_t} K_i$ is a compound Poisson jump process, the Poisson process $dP(t)$ with constant intensity λ causes the price to jump randomly and the mean of the relative price jump is $\bar{k} \equiv E(k_i - 1) = e^{\gamma + \frac{\delta^2}{2}} - 1$ with variance $E([k_i - 1 - E(k_i - 1)^2]) = e^{2\gamma+\delta^2}(e^{\delta^2} - 1)$, [21].

The model defined in equation (3) is the Merton Jump process is an extension of the Black-Scholes model to cater for real life assumptions including market returns do not follow a constant variance log-normal distribution. The model comprises two parts. The first part is a diffusion part with a standard wiener process W(t), while the second part is made up of jump with a compound Poisson process $\sum_{i=1}^{L_t} K_i$. The addition of a compound Poisson process is what differentiates the Merton model [17] from Black-Scholes model [4]. A compound Poisson process is a stochastic process with jumps, since occurrence of events and random arrival time are taken as jumps, which are random, hence they follow a Poisson process.

2.3 Proposed Jump Process Model

Our proposed model extends the Merton specification in equation (3) by adding compound Poisson process with nonconstant intensity, that is, we posit that the average number of jumps per unit time is a function of time and this is from the argument that in real life application, a dynamical system is influenced by time because a lot of factors would influence the system in the course of time. In addition to capturing the negative skewness and excess kurtosis of the log return density [17], we assume that the random jumps follow a Poisson process characterized by its intensity function $\eta(t) = \int_0^t \lambda(s) ds$ and which we note to have the distribution

$$\mathcal{P}(P(t) = r) = e^{-\eta(t)} \frac{\eta(t)^r}{r!}, \qquad r = 0,1,2, \ldots$$

where the intensity is function of time $\lambda = \lambda(t)$.

For the sake of reference, we shall refer to our proposed model as extended Merton jump nonconstant intensity diffusion (xMJNID) model. We proceed thus, suppose in the small time interval the asset price jumps from $X(t)$ to $kX(t)$. So the percentage change in the asset price caused by the jump is

$$\frac{dX(t)}{X(t)} = \frac{kX(t) - X(t)}{X(t)} = k - 1$$

where $\ln(k) \sim i.i.d. N(\gamma, \delta^2)$ and means that

$$E(k) = e^{\gamma+\frac{\delta^2}{2}} \quad \text{and} \quad E([k - E(k)^2]) = e^{2\gamma+\delta^2}(e^{\delta^2} - 1)$$

Since if $\ln x \sim N(a, b)$, then $x \sim \log \text{normal}\ (e^{a+\frac{b^2}{2}}, e^{2a+b^2}(e^{b^2} - 1))$.

So the SDE takes the form

$$\frac{dX(t)}{X(t)} = (\alpha - (b + \lambda t)\ k)dt + \sigma dW(t) + (k - 1)dP(t) \tag{4}$$

where α is the instantaneous expected return on the asset, σ is the instantaneous volatility of the asset return conditional on that jump does not occur, $W(t)$ is a standard Brownian motion process, and $P(t)$ is an Poisson process with nonconstant intensity $\lambda_t = b + \lambda t$. Also, the Brownian motion, the lognormal jumps, and the Poisson process are assumed to be independent. The relative price jump size of $X(t)$, $k - 1$, is lognormally distributed with the mean and variance given respectively as

$$E(k - 1) = e^{\gamma+\frac{\delta^2}{2}} - 1 \equiv \bar{k} \quad \text{and} \quad E[k - 1 - E(k - 1)^2] = e^{2\gamma+\delta^2}(e^{\delta^2} - 1).$$

The expected relative rate change $E\left(\frac{dX(t)}{X(t)}\right)$ from the jump part $dP(t)$ in the time interval dt is $(b + \lambda t)\ \bar{k}dt$. This is why the instantaneous expected return on the asset αdt is adjusted by $-(b + \lambda t)\ \bar{k}dt$ in the drift term of the jump-diffusion process to make the jump part an unpredictable innovation

$$E\left(\frac{dX(t)}{X(t)}\right) = E\left[\left(\alpha - (b + \lambda t)\ \bar{k}\right)dt\right] + E[\sigma dW(t)] + E[(k - 1)dP(t)]$$
$$= \left(\alpha - (b + \lambda t)\ \bar{k}\right)dt + 0 + (b + \lambda t)\ \bar{k}dt = \alpha dt$$

From equation (4),

$$dX(t) = (\alpha - (b + \lambda t)\ k)X(t)dt + \sigma X(t)dW(t) + (k - 1)X(t)dP(t) \tag{5}$$

[19] give the Itô formula for the jump-diffusion process as

$$df(X(t), t) = \frac{\partial f(X(t), t)}{\partial t}dt + b(t)\frac{\partial f(X(t), t)}{\partial x}dt + \frac{\sigma_t^2}{2}\frac{\partial^2 f(X(t), t)}{\partial x^2}dt$$
$$+ \sigma(t)\frac{\partial f(X(t), t)}{\partial x}dW(t) + [f(X(t-) + \Delta X(t)) - f(X(t-))],$$

where $b(t)$ corresponds to the drift term and $\sigma(t)$ corresponds to the volatility term of a jump-diffusion process

$$X(t) = X(0) + \int_0^t b(s)ds + \int_0^t \sigma(s)dW(s) + \sum_{i=1}^{P(t)} \Delta X(i)$$

Following same we have

$$d\ln X(t) = \frac{\partial \ln X(t)}{\partial t}dt + \left(\alpha - (b + \lambda t)\ \bar{k}\right)X(t)\frac{\partial \ln X(t)}{\partial X(t)} + \frac{\sigma^2(X(t))^2}{2}\frac{\partial^2 \ln X(t)}{\partial (X(t))^2}dt$$
$$+ \sigma X(t)\frac{\partial \ln X(t)}{\partial X(t)}dW(t) + [\ln kX(t) - \ln X(t)] \tag{6}$$

$$d\ln X(t) = \left(\alpha - (b + \lambda t)\ \bar{k}\right)X(t)\frac{1}{X(t)}dt + \frac{\sigma^2(X(t))^2}{2}\left(-\frac{1}{(X(t))^2}\right)dt + \sigma X(t)\frac{1}{X(t)}dW(t)$$
$$+ [\ln k + \ln X(t) - \ln X(t)]$$
$$= \left(\alpha - (b + \lambda t)\ \bar{k}\right)dt - \frac{\sigma^2}{2}dt + \sigma dW(t) + \ln k$$

$$\ln X(t)-\ln X(0)=\left(\alpha-\frac{\sigma^2}{2}(b+\lambda t)\,\bar{k}\right)(t-0)+\sigma(t)(W(t)-W(0))+\sum_{i=1}^{P(t)}\ln k_i$$

$$\ln X(t)=\ln X(0)+\left(\alpha-\frac{\sigma^2}{2}(b+\lambda t)\,\bar{k}\right)(t-0)+\sigma(t)(W(t)-W(0))+\sum_{i=1}^{P(t)}\ln k_i$$

$$\ln X(t)=\ln X(0)+\left(\alpha-\frac{\sigma^2}{2}(b+\lambda t)\,\bar{k}\right)(t-0)+\sigma(t)W(t)+\sum_{i=1}^{P(t)}\ln k_i$$

$$\exp(\ln X(t))=\exp\left\{\ln X(0)+\left(\alpha-\frac{\sigma^2}{2}(b+\lambda t)\,\bar{k}\right)(t-0)+\sigma(t)W(t)+\sum_{i=1}^{P(t)}\ln k_i\right\}$$

$$X(t)=X(0)\exp\left\{\left(\alpha-\frac{\sigma^2}{2}(b+\lambda t)\,\bar{k}\right)t+\sigma(t)W(t)\right\}\exp\left(\sum_{i=1}^{P(t)}\ln k_i\right)$$

$$X(t)=X(0)\exp\left\{\left(\alpha-\frac{\sigma^2}{2}(b+\lambda t)\,\bar{k}\right)t+\sigma W(t)\right\}\prod_{i=1}^{P(t)}k_i$$

$$X(t)=X(0)\exp\left\{\left(\alpha-\frac{\sigma^2}{2}(b+\lambda t)\,\bar{k}\right)t+\sigma W(t)+\sum_{i=1}^{P(t)}\ln k_i\right\} \tag{7}$$

Using the previous definition of the log rate jump size $\ln k_i = K_i$

$$X(t)=X(0)\exp\left\{\left(\alpha-\frac{\sigma^2}{2}(b+\lambda t)\,\bar{k}\right)t+\sigma W(t)+\sum_{i=1}^{P(t)}K_i\right\} \tag{8}$$

This implies that $X(t)$ is an exponential Lévy model $X(t)=X(0)e^{P(t)}$ with a compound Poisson jump part given as

$$CP(t)=\left(\alpha-\frac{\sigma^2}{2}(b+\lambda t)\,\bar{k}\right)t+\sigma W(t)+\sum_{i=1}^{P(t)}K_i$$

We note that the compound Poisson jump process $\prod_{i=1}^{P(t)}k_i=1$ if $P(t)=0$ or positive and negative jumps cancel each other out.

In the Black-Scholes case, log return $\ln(X(t)/X(0))$ is normally distributed [4]

$$X(t)=X(0)\exp\left\{\left(\alpha-\frac{\sigma^2}{2}\right)t+\sigma W(t)\right\}$$

$$\ln(X(t)/X(0))\sim N\left(\left(\alpha-\frac{\sigma^2}{2}\right)t,\sigma^2 t\right)$$

[17] posited that the existence of compound Poisson jump process makes log return non-normal, which enables the probability density of log return $y(t)=\ln(X(t)/X(0))$ to be obtained as a quickly converging series of the following form

$$\mathcal{P}(y(t)\in A)=\sum_{i=0}^{\infty}\mathcal{P}(P(t)=i)\mathcal{P}(y(t)\in A|P(t)=i)$$

$$\mathcal{P}(y(t))=\sum_{i=0}^{\infty}\frac{e^{-\lambda t}(\lambda t)^i}{i!}N\left(y(t);\left(\alpha-\frac{\sigma^2}{2}(b+\lambda t)\,\bar{k}\right)t+i\gamma,\sigma^2 t+i\delta^2\right) \tag{9}$$

where

$$N\left(y(t);\left(\alpha-\frac{\sigma^2}{2}(b+\lambda t)\,\bar{k}\right)t+i\gamma,\sigma^2 t+i\delta^2\right)$$

$$=\frac{1}{\sqrt{2\pi\sigma^2 t+i\delta^2}}\exp\left[-\frac{x_t-\left\{\left(\alpha-\frac{\sigma^2}{2}(b+\lambda t)\,\bar{k}\right)t+i\gamma\right\}}{2(\sigma^2 t+i\delta^2)}\right]$$

The term $\mathcal{P}(P(t)=i)=\frac{e^{-\lambda t}(\lambda t)^i}{i!}$ is the probability that the asset price jumps i times during the time interval of length t and

$$\mathcal{P}(y(t)\in A|P(t)=i)=N\left(x_t;\left(\alpha-\frac{\sigma^2}{2}(b+\lambda t)\,\bar{k}\right)t+i\gamma,\sigma^2 t+i\delta^2\right)$$

is the Black-Scholes normal density of log-return assuming that the asset price jumps i times in the time interval of t. Therefore, the log-return density as in the Merton model can be interpreted as the weighted average of the Black-Scholes normal density by the probability that the asset price jumps i times.

The characteristic function of the model can be calculated by Fourier transform of the log-return density function with parameters $(a,b)=(1,1)$

$$\phi(\omega)=\int_{-\infty}^{\infty}\exp(i\omega y(t))\,\mathcal{P}(y(t))dy(t) \tag{10}$$

$$=\exp\left[(b+\lambda t)\exp\left\{\frac{1}{2}\omega(2i\gamma-\delta^2\omega)\right\}-(b+\lambda t)\big(1+i\omega\bar{k}\big)-\frac{1}{2}t\omega\{-2i\alpha+\sigma^2(i+\omega)\}\right]$$

$$=\exp\left[(b+\lambda t)\exp\{\omega i\gamma-\delta^2\omega^2\}-\big(b+bi\omega\bar{k}+\lambda t+i\omega\bar{k}\lambda t\big)+i\alpha t\omega-\frac{1}{2}t\omega\sigma^2 i-\frac{1}{2}ti\omega^2\right]$$

$$=\exp\left[(b+\lambda t)\exp\{\omega i\gamma-\delta^2\omega^2\}-b-bi\omega\bar{k}-\lambda t-i\omega\bar{k}\lambda t+i\alpha t\omega-\frac{1}{2}t\omega\sigma^2 i-\frac{1}{2}ti\omega^2\right]$$

$$=\exp\left[(b+\lambda t)\exp\{\omega i\gamma-\delta^2\omega^2\}-\frac{i\omega}{2}\big(2b\bar{k}+2\bar{k}\lambda t-2\alpha t-t\sigma^2\big)-\frac{1}{2}ti\omega^2-(b+\lambda t)\right]$$

$$=\exp\left[(b+\lambda t)\exp\{\omega i\gamma-\delta^2\omega^2\}-(b+\lambda t)-\frac{i\omega}{2}\big(2b\bar{k}+2\bar{k}\lambda t-2\alpha t-t\sigma^2\big)-\frac{1}{2}ti\omega^2\right]$$

$$=\exp\left[(b+\lambda t)\exp(\{\omega i\gamma-\delta^2\omega^2\}-1)-\frac{i\omega}{2}\big(2b\bar{k}+2\bar{k}\lambda t-2\alpha t-t\sigma^2\big)-\frac{1}{2}ti\omega^2\right]$$

$$=\exp\left[(b+\lambda t)\exp(\{\omega i\gamma-\delta^2\omega^2\}-1)-i\omega\left(b\bar{k}+\bar{k}\lambda t-\alpha t-\frac{t\sigma^2}{2}\right)-\frac{1}{2}ti\omega^2\right]$$

$$=\exp\left[(b+\lambda t)\exp(\{\omega i\gamma-\delta^2\omega^2\}-1)-i\omega\left(-\alpha t-\frac{t\sigma^2}{2}+(b+\lambda t)\bar{k}\right)-\frac{1}{2}ti\omega^2\right]$$

$$=\exp\left[(b+\lambda t)\exp(\{\omega i\gamma-\delta^2\omega^2\}-1)+i\omega\left(\alpha t-\frac{t\sigma^2}{2}(b+\lambda t)\,\bar{k}\right)-\frac{1}{2}ti\omega^2\right]$$

Let

$$\psi(\omega)=(b+\lambda t)\exp(\{\omega i\gamma-\delta^2\omega^2\}-1)+i\omega\left(\alpha t-\frac{t\sigma^2}{2}(b+\lambda t)\,\bar{k}\right)-\frac{1}{2}ti\omega^2 \tag{11}$$

be the characteristic exponent or cumulant generating function, where $\bar{k}\equiv e^{\gamma+\frac{\delta^2}{2}}-1$. We then have

$$\phi(\omega)=\exp[t\psi(\omega)]$$

Following the procedure of [21], the characteristic exponent (11) can be alternatively obtained by substituting the Lévy measure of the model

$$\ell(dx) = \frac{\lambda}{\sqrt{2\pi\delta^2}} \exp\left\{\frac{(dx-\gamma)}{2\delta^2}\right\} = (b+\lambda t) f(dx)$$

into the Lévy-Khinchin representation of the finite variation type

$$\psi(\omega) = \omega i v - \frac{\delta^2\omega^2}{2} + \int_{-\infty}^{\infty} \{\exp(i\omega x) - 1\} \ell(dx)$$

$$\psi(\omega) = \omega i v - \frac{\delta^2\omega^2}{2} + \int_{-\infty}^{\infty} \{\exp(i\omega x) - 1\} (b+\lambda t) f(dx)$$

$$\psi(\omega) = \omega i v - \frac{\delta^2\omega^2}{2} + (b+\lambda t) \int_{-\infty}^{\infty} \{\exp(i\omega x) - 1\} f(dx)$$

$$\psi(\omega) = \omega i v - \frac{\delta^2\omega^2}{2} + (b+\lambda t) \left\{\int_{-\infty}^{\infty} \exp(i\omega x) f(dx) - \int_{-\infty}^{\infty} f(dx)\right\} \tag{12}$$

Since $\int_{-\infty}^{\infty} \exp(i\omega x) f(dx)$ is the characteristic function of $f(dx)$,

$$\int_{-\infty}^{\infty} \exp(i\omega x) f(dx) = \exp\left(\omega i v - \frac{\delta^2\omega^2}{2}\right)$$

Therefore,

$$\psi(\omega) = \omega i v - \frac{\delta^2\omega^2}{2} + (b+\lambda t)\left\{\exp\left(\omega i v - \frac{\delta^2\omega^2}{2}\right) - 1\right\} \tag{13}$$

where $v = \alpha t - \frac{t\sigma^2}{2}(b+\lambda t)\,\bar{k}$. This corresponds to (11).

2.3.1 Log Exchange rate process with the nonconstant intensity

We recall the log exchange rate dynamics as

$$\ln X(t) = \ln X(0) + \left(\alpha - \frac{\sigma^2}{2}(b+\lambda t)\,\bar{k}\right)(t-0) + \sigma(t)W(t) + \sum_{i=1}^{L_t} \ln k_i$$

Hence the characteristic function of the probability density of log exchange rate $\ln X(t)$ in equation (10) with the term $y(t)$ replaced by $\ln X(t)$ and with the Fourier transform above is

$$\phi(\omega) = \int_{-\infty}^{\infty} \exp(i\omega \ln X(t))\, \mathcal{P}(\ln X(t)) d \ \ln X(t)$$

$$= \exp\left[(b+\lambda t)\exp(\{\omega i\gamma - \delta^2\omega^2\} - 1) + i\omega\left(\alpha t - \frac{t\sigma^2}{2}(b+\lambda t)\,\bar{k}\right) - \frac{1}{2} t i\omega^2\right] \tag{14}$$

with $\bar{k} = e^{\gamma + \frac{\delta^2}{2}} - 1$.

The diffusion process of the log-return for the exchange rate is

$$X(t) = X(0) e^{\left(r-(b+\lambda t)\bar{k} - \frac{\sigma^2}{2}\right)t + \sigma W_t} U\big(n(t)\big) \tag{15}$$

where

$$U\big(n(t)\big) = \prod_{i=0}^{n(t)} (1+k_i)$$

2.3.2 Derivation of the arbitrage-free dynamics of $X(t)$

We introduce the auxiliary process $X(t)^*$ given by

$$X(t)^* := \frac{W(t)^f X(t)}{W(t)^d} = e^{(r_f - r_d)t} X(t) \tag{16}$$

Hence,

$$X(t)^* = X(0) \exp\left(\left(\alpha - \frac{\sigma^2}{2}(b + \lambda t)\,\bar{k}\right) t + \sigma W(t) + P(t)\right) \tag{17}$$

where $\tilde{\mu} = \mu + r_f - r_d$.

Since $P(t)$ is a subordinator, we have

$$P(t) = mt + \int_0^t \int_0^\infty zN(ds, dz) = \int_0^t \int_0^\infty zN(ds, dz), \quad m = 0$$

The last equation follows from

$$E[P(t)] = mt + E\left[\int_0^t \int_0^\infty zN(ds, dz)\right] = mt + E\left[\int_0^t \int_0^\infty zN(ds, dz)\right]$$

$$= mt + E\left[t\int_0^\infty zv(dz)\right] = mt + E\left[t\int_0^\infty zaz^{-1}e^{-bz}dz\right]$$

$$= mt + E\left[at\int_0^\infty e^{-bz}\right] = mt + E\left[at\frac{1}{b}\right] = mt + t\frac{a}{b}$$

Since the expectation of the Gamma process $E[P(t)]$ equals $\frac{a}{b}$, we have to have $m = 0$. By the Itô formula we proceed with the following mathematical computation.

$$X(t)^* = X(0) + \int_0^t X(s)^* \sigma dW(s) + \int_0^t X(s)^* \left(\tilde{\mu} - \frac{1}{2}\sigma^2\right) ds + \frac{1}{2}\int_0^t X(s)^* \sigma^2 ds$$
$$+ \int_0^t \int_0^\infty X(s-)^* (e^z - 1) N(ds, dz)$$
$$= X(0) + \int_0^t X(s)^* \sigma dW(s) + \int_0^t X(s)^* \tilde{\mu} ds + \int_0^t \int_0^\infty X(s)^* (e^z - 1) v(dz) ds$$
$$+ \int_0^t \int_0^\infty X(s-)^* (e^z - 1) \widetilde{N}(ds, dz)$$
$$= X(0) + \int_0^t X(s)^* \sigma dW(s) + \int_0^t X(s)^* \tilde{\mu} ds + \int_0^t \int_0^\infty X(s)^* (e^z - 1) v(dz) ds$$
$$+ \int_0^t \int_0^\infty X(s-)^* (e^z - 1) \widetilde{N}(ds, dz),$$

where we have inserted the compensated Poisson measure $\widetilde{N}(dz, ds) = N(dz, ds) - v(dz)ds$ in the second equality.

We see that if

$$\tilde{\mu} + \int_0^\infty (e^z - 1)v(dz) = 0,$$

then $X(t)^*$ is a local martingale with respect to $P^* = P$ (the physical measure). Hence, $X(t)$ has arbitrage-free dynamics if

$$\mu = r_d - r_f - \int_0^\infty (e^z - 1)v(dz). \tag{18}$$

2.3.3 General risk neutral measures for the xMJNID model

We note that to have $a, b > 0$, then $\int_0^\infty (e^z - 1)^2 v(dz)$ exists. This is so because, by insertion of the Lévy measure, we have that

$$\int_0^\infty (e^z - 1)^2 v(dz) = \int_0^\infty (e^z - 1)^2 az^{-1}e^{-bz}dz, \quad \text{when } v(dz) = az^{-1}e^{-bz}$$

Moreover, we can see that using the mean value theorem, $e^z - 1 = e^z - e^0 = z \cdot \int_0^1 e^{\theta z} d\theta$, we have that

$$\int_0^\infty (e^z - 1)v(dz) = \int_0^\infty \int_0^1 e^{\theta z} d\theta \, z \, v(dz) \tag{19}$$

and inputting the expression for the Lévy measure, we have

$$= \int_0^\infty \int_0^1 e^{\theta z} d\theta \cdot ae^{-bz}dz = a\int_0^1 \left(\int_0^\infty e^{z(\theta - b)}dz \right) d\theta$$

where we utilize Fubini's theorem. Then we have,

$$= a\int_0^1 \left[\frac{1}{\theta - b} e^{z(\theta-b)} \Big|_0^\infty \right] d\theta = a\int_0^1 \frac{1}{\theta - b} d\theta$$

where also we assume that $b > 1$.

$$= a \cdot [\log(b - \theta)|_0^1] = a(\log(b-1) - \log(b)).$$

The log-returns are given by

$$Y(t_i) = \log\left(\frac{X(t_i)}{X(t_{i-1})}\right) \tag{20}$$

$$= \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\big(W(t_i) - W(t_{i-1})\big) + P(t_i) - P(t_{i-1})$$

$$= \left(r_d - r_f - a(\log(b-1) - \log(b)) - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\big(W(t_i) - WB(t_{i-1})\big) + P(t_i) - P(t_{i-1})$$

Here $W(t_i)$ and $P(t_i)$ are independent stochastic processes and $Y(t_i)$ can be written in the following way:

$$Y(t_i) = y_1 + y_2$$

where

$$y_1 := \left(r_d - r_f - a(\log(b-1) - \log(b)) - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\big(W(t_i) - W(t_{i-1})\big)$$

$$y_2 := P(t_i) - P(t_{i-1})$$

In order to find the density of $Y(t_i)$ we compute

$$\mathcal{P}(Y(t_i) \le y) = \mathcal{P}(y_1 + y_2 \le x) = E\left[\mathbb{1}_{\{y_1 + y_2 \le y\}}(y_1 + y_2)\right] \tag{21}$$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty} \mathbb{1}_{\{y_1+y_2\le y\}}(y_1+y_2)\cdot f_{Y_1,Y_2}^{(y_1,y_2)}dy_1dy_2$$

with y_1, y_2 independent, then

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} \mathbb{1}_{\{y_1+y_2\le y\}}(y_1+y_2)\cdot f_{Y_1,Y_2}^{(y_1,y_2)}dy_1dy_2$$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty} \mathbb{1}_{(-\infty,y]}(y_1)\mathbb{1}_{[0,y-y_1]}(y_2)\cdot f_{Y_1}^{(y_1)}f_{Y_2}^{(y_2)}dy_1dy_2$$

$$= \int_{-\infty}^{y}\int_{0}^{y-y_1} f_{Y_1}^{(y_1)}f_{Y_2}^{(y_2)}dy_2dy_1,$$

where $\mathbb{1}_{(-\infty,y)}$ is the indicator function of the interval $(-\infty, y]$.
The density of the log-returns are now given by

$$f_{Y(t_i)}(y) = \frac{\partial}{\partial y}P(Y(t_i)\le y) = \int_{-\infty}^{y} f_{Y_1}(y_1)f_{Y_2}(y-y_1)dy_1, \tag{22}$$

where

$$f_{Y_1}(y) = \frac{1}{(2\pi\Delta t\sigma^2)^{\frac{1}{2}}}\exp\left(-\frac{\left(y-\left(r_d-r_f-a(\log(b-1)-\log(b))-\frac{1}{2}\sigma^2\right)\Delta t\right)^2}{2\Delta t\sigma^2}\right)$$

$$f_{Y_2}(y) = \frac{b^{a\Delta t}}{\Gamma(a\Delta t)}x^{a\Delta t-1}e^{-by}.$$

Hence,

$$f_{Y_1}(y_1)f_{Y_2}(y-y_1)$$

$$= \frac{1}{(2\pi\Delta t\sigma^2)^{\frac{1}{2}}}\exp\left(-\frac{\left(y_1-\left(r_d-r_f-a(\log(b-1)-\log(b))-\frac{1}{2}\sigma^2\right)\Delta t\right)^2}{2\Delta t\sigma^2}\right)$$

$$\cdot\frac{b^{a\Delta t}}{\Gamma(a\Delta t)}(y-y_1)^{a\Delta t-1}e^{-b(y-y_1)}$$

$$= \frac{1}{(2\pi\Delta t\sigma^2)^{\frac{1}{2}}}\exp\left(-\frac{(y_1-s)^2}{2\Delta t\sigma^2}\right)\cdot\frac{b^{a\Delta t}}{\Gamma(a\Delta t)}(x-y_1)^{a\Delta t-1}e^{by_1}e^{-by}$$

$$= \frac{1}{(2\pi\Delta t\sigma^2)^{\frac{1}{2}}}\exp\left(-\frac{(y_1-s)^2}{2\Delta t\sigma^2}+by_1\right)\cdot\frac{b^{a\Delta t}}{\Gamma(a\Delta t)}(x-y_1)^{a\Delta t-1}e^{-by}, \tag{23}$$

where we have defined $s := \left(r_d-r_f-a(\log(b-1)-\log(b))-\frac{1}{2}\sigma^2\right)\Delta t$. Taking the expression in the exponent in (22), we have that

$$-\frac{(y_1-s)^2}{2\Delta t\sigma^2}+by_1 = -\frac{1}{2\Delta t\sigma^2}(y_1^2-2(s+\Delta t\sigma^2 b)y_1)+s^2)$$

$$= -\frac{1}{2\Delta t\sigma^2}(y_1^2-2\tilde{s}y_1+s^2)$$

$$= -\frac{1}{2\Delta t\sigma^2}(y_1^2-2\tilde{s}y_1+\tilde{s}^2-\tilde{s}^2+s^2)$$

$$= -\frac{1}{2\Delta t\sigma^2}((y_1 - \tilde{s})^2 - \tilde{s}^2 + s^2)$$
$$= -\frac{(y_1 - \tilde{s})^2}{2\Delta t\sigma^2} + \frac{(-\tilde{s}^2 + s^2)}{2\Delta t\sigma^2}$$

where we have defined $\tilde{s} := s + \Delta t\sigma^2 b$.
The likelihood function is given by

$$L(y_1, \dots, y_m; \sigma^2, a, b) = \prod_{i=1}^{m} f_{Y(t_i)}(y_i) \tag{24}$$

and we find the MLE's by considering the equations

$$\frac{\partial}{\partial\sigma^2} L(y_1, \dots, y_m; \sigma^2, a, b) = 0 \tag{25}$$
$$\frac{\partial}{\partial a} L(y_1, \dots, y_m; \sigma^2, a, b) = 0 \tag{26}$$
$$\frac{\partial}{\partial b} L(y_1, \dots, y_m; \sigma^2, a, b) = 0 \tag{27}$$

We will for simplicity assume that $a\Delta t = 1$, consequently $a = \frac{1}{\Delta t}$ and the calculation of (26) is omitted. Insertion of (23) into (22) gives

$$f_{Y(t_i)}(x) = \int_{-\infty}^{y} \frac{1}{(2\pi\Delta t\sigma^2)^{\frac{1}{2}}} \exp\left(-\frac{(y_1 - \tilde{s})^2}{2\Delta t\sigma^2} + \frac{(-\tilde{s}^2 + s^2)}{2\Delta t\sigma^2}\right) \cdot \frac{b^{a\Delta t}}{\Gamma(a\Delta t)}(x - x_1)^{a\Delta t - 1} b e^{-by} dy_1,$$

with $a\Delta t = 1$, then it becomes

$$= \int_{-\infty}^{y} \frac{1}{(2\pi\Delta t\sigma^2)^{\frac{1}{2}}} \exp\left(-\frac{(y_1 - \tilde{s})^2}{2\Delta t\sigma^2} + \frac{(-\tilde{s}^2 + s^2)}{2\Delta t\sigma^2}\right) \cdot b e^{-by} dy_1$$
$$= \exp\left(-\frac{s^2 - \tilde{s}^2}{2\Delta t\sigma^2}\right) \cdot b e^{-by} \cdot \frac{1}{(2\pi\Delta t\sigma^2)^{\frac{1}{2}}} \int_{-\infty}^{y} \exp\left(-\frac{(y_1 - \tilde{s})^2}{2\Delta t\sigma^2}\right) dy_1$$
$$= \exp\left(-\frac{s^2 - \tilde{s}^2}{2\Delta t\sigma^2}\right) b e^{-by} \cdot \Phi_{\tilde{s},\Delta t\sigma^2}(y_i), \tag{28}$$

where $\Phi_{\tilde{s},\Delta t\sigma^2}(y_i)$ is the normal cumulative distribution with mean $\tilde{s}$ and variance $\Delta t\sigma^2$.
The likelihood function can now be written as

$$L(y_1, \dots, y_m; \sigma^2, b) = \prod_{i=1}^{m} f_{X(t_i)}(x_i)$$
$$= b^m \exp\left(-m\frac{(s^2 - \tilde{s}^2)}{2\Delta t\sigma^2}\right) \exp\left(-b\sum_{i=1}^{m} y_i\right) \cdot \prod_{i=1}^{m} \Phi_{\tilde{s},\Delta t\sigma^2}(y_i). \tag{29}$$

2.4 Estimation of log-returns for the $X(t)$ with xMJNID model

The estimation approach to be used here in based on quasi-likelihood method. From the log-likelihood function of equation (29), we choose the physical measure, i.e. $u(t) = 0$ in the Girsanov calculations, resulting in μ as the constant $= r_d - r_f$, and also take we $b = 1$ and have the following likelihood function

$$L(y_1, \dots, y_m; \sigma^2) = \log \prod_{i=1}^{m} f_{Y(t_i)}(y_i) = \frac{1}{(2\pi\Delta t\sigma^2)^{\frac{m}{2}}} \exp\left(-\frac{\sum_{i=1}^{m}\left(y - \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t\right)^2}{2\Delta t\sigma^2}\right) \tag{30}$$

and the log-likelihood function

$$l(y_1,\dots,y_m;\sigma^2)=\log[L(y_1,\dots,y_m;\sigma^2;b)]=\log\left(\frac{1}{(2\pi\Delta t\sigma^2)^{\frac{m}{2}}}\right)-\frac{\sum_{i=1}^{m}\left(y-\left(\mu-\frac{1}{2}\sigma^2\right)\Delta t\right)^2}{2\Delta t\sigma^2}$$

$$=-\frac{m}{2}\log(2\pi\Delta t\sigma^2)-\frac{\sum_{i=1}^{m}\left(y-\left(\mu-\frac{1}{2}\sigma^2\right)\Delta t\right)^2}{2\Delta t\sigma^2}$$

Differentiating with respect to σ^2 and equating to zero gives

$$\frac{\partial}{\partial\sigma^2}l(y_1,\dots,y_m;\sigma^2)$$

$$=-\frac{m}{2}\frac{1}{2\pi\Delta t\sigma^2}\cdot(2\pi\Delta t)$$

$$-\frac{2\sum_{i=1}^{m}\left(y_i-\left(\mu-\frac{1}{2}\sigma^2\right)\Delta t\right)\cdot\frac{1}{2}\Delta t\cdot 2\Delta t\sigma^2-\sum_{i=1}^{m}\left(y_i-\left(\mu-\frac{1}{2}\sigma^2\right)\Delta t\right)^2\cdot 2\Delta t}{(2\Delta t\sigma^2)^2} \qquad (31)$$

$$=-\frac{m}{2\sigma^2}-\frac{\sum_{i=1}^{m}\left(y_i-\left(\mu-\frac{1}{2}\sigma^2\right)\Delta t\right)\cdot\Delta t\sigma^2-\sum_{i=1}^{m}\left(y_i-\left(\mu-\frac{1}{2}\sigma^2\right)\Delta t\right)^2}{(2\Delta t\sigma^4)}$$

$$=\frac{-m\Delta t\sigma^2-\sum_{i=1}^{m}\left(y_i-\left(\mu-\frac{1}{2}\sigma^2\right)\Delta t\right)\cdot\Delta t\sigma^2+\sum_{i=1}^{m}\left(y_i-\left(\mu-\frac{1}{2}\sigma^2\right)\Delta t\right)^2}{(2\Delta t\sigma^4)}=0$$

and solving for σ^2 gives the quasi-maximum likelihood estimate

$$-m\Delta t\sigma^2-\sum_{i=1}^{m}\left(y_i-\left(\mu-\frac{1}{2}\sigma^2\right)\Delta t\right)\cdot\Delta t\sigma^2+\sum_{i=1}^{m}\left(y_i-\left(\mu-\frac{1}{2}\sigma^2\right)\Delta t\right)^2=0$$

$$-m\Delta t\sigma^2-\Delta t\sigma^2\sum_{i=1}^{m}\left(y_i-\left(\mu-\frac{1}{2}\sigma^2\right)\Delta t\right)+\sum_{i=1}^{m}\left(y_i^2-2y_i\left(\mu-\frac{1}{2}\sigma^2\right)\Delta t\right)\left(\left(\mu-\frac{1}{2}\sigma^2\right)\Delta t\right)^2$$
$$=0$$

$$-m\Delta t\sigma^2-\Delta t\sigma^2\sum_{i=1}^{m}x_i+m(\Delta t)^2\sigma^2\left(\mu-\frac{1}{2}\sigma^2\right)+\sum_{i=1}^{m}y_i^2-2\Delta t\left(\mu-\frac{1}{2}\sigma^2\right)\sum_{i=1}^{m}y_i$$
$$+m\left(\mu-\frac{1}{2}\sigma^2\right)^2(\Delta t)^2=0$$

$$-m\Delta t\sigma^2-\Delta t\sigma^2\sum_{i=1}^{m}y_i+m(\Delta t)^2\sigma^2\left(\mu-\frac{1}{2}\sigma^2\right)+\sum_{i=1}^{m}y_i^2-2\Delta t\left(\mu-\frac{1}{2}\sigma^2\right)\sum_{i=1}^{m}y_i$$
$$+m\left(\mu-\frac{1}{2}\sigma^2\right)^2(\Delta t)^2=0$$

$$-m\Delta t\sigma^2+\sum_{i=1}^{m}y_i^2-2\Delta t\mu\sum_{i=1}^{m}y_i-\frac{m}{4}\sigma^2(\Delta t)^2+m(\Delta t)^2\mu^2=0$$

$$\frac{m}{4}\sigma^4(\Delta t)^2+m\Delta t\sigma^2-\left(\sum_{i=1}^{m}y_i^2-2\Delta t\mu\sum_{i=1}^{m}y_i+m(\Delta t)^2\mu^2\right)=0$$

$$\frac{m}{4}\sigma^4(\Delta t)^2+m\Delta t\sigma^2-\left(\sum_{i=1}^{m}(y_i-\Delta t\mu)^1\right)=0$$

This is an equation of second order with respect to σ^2

$$\sigma^2 = \frac{-m\Delta t \pm \sqrt{(m\Delta t)^2 - 4 \cdot \frac{m}{4}(\Delta t)^2 \cdot (-1)\sum_{i=1}^{m}(y_i - \Delta t\mu)^2}}{2 \cdot \frac{m}{4}(\Delta t)^2}$$

$$\sigma^2 = \frac{-m\Delta t \pm \sqrt{(m\Delta t)^2 + m(\Delta t)^2 \sum_{i=1}^{m}(y_i - \Delta t\mu)^2}}{\frac{m}{2}(\Delta t)^2}$$

$$\sigma^2 = \frac{-m\Delta t \pm m\Delta t\sqrt{(1 + m^{-1}\sum_{i=1}^{m}(y_i - \Delta t\mu)^2}}{\frac{m}{2}(\Delta t)^2}$$

$$\hat{\sigma}^2 = \frac{-1 \pm \sqrt{1 + m^{-1}\sum_{i=1}^{m}(y_i - \Delta t\mu)^2}}{\frac{1}{2}\Delta t} \qquad (32)$$

The diffusion coefficient quasi-maximum likelihood estimate is given as (32).

Results and discussion

3.1 Simulation study results

This section presents the results of the simulation study comparing the xMJNID model to existing models and evaluating their performances using model diagnostics, Akaike information criterion (AIC) and Bayesian information criterion (BIC), while root mean squared error (RMSE), mean absolute scaled error (MASE) and symmetric mean absolute percentage error (SMAPE) were used to compare the parameter estimates to the specified true values.

We simulated levy jumps using compound Passion process setting the parameters' values as alpha = 1, lambda = 0.5, drift = 0, diffusion = 2 and Gaussian distribution with mean 0.01 and standard deviation 0.1, for observation sizes 200, 500, 4500 and 10000. We fitted all the models in consideration and present the results in Table 1 and Table 2.

Table 1: Comparison of models with Simulated data for N = 200 and 500

			N = 200			
model	GBm	O-U	VAS	CEV	Merton	xMJNID
drift	4.19794	3.36979	0.00000	2.65734	4.19794	4.19791
diffusion	2.45927	32.82588	1712.17500	2.00000	2.44496	2.44495
mase	1.164302	8.548916			1.160725	1.160714
rmse	2.986103	21.92704	1710.175	0.789956	2.985019	2.984995
smape	1.102992	1.885143	1.995333	1.740004	1.100105	1.100103
AIC	656.4956	860.87	2245.971	959.0319	662.4979	588.9815
BIC	663.0923	867.4666	2255.866	968.9269	678.9895	608.7714
			N = 500			
drift	1.52022	1.19079	0.00000	1.00000	1.15266	1.15262
diffusion	1.15266	13.84775	0.00001	2.00000	1.51848	1.15282
mase	0.40811	3.25964			0.40854	0.40859
rmse	0.88284	8.41983	3801.48	0.90183	0.88331	0.88342
smape	1.13629	1.74760	1.99790	1.66715	1.13685	1.13686
AIC	1073.80	1285.17	6403.90	1110.49	1079.83	297.22
BIC	1082.22	1293.59	6416.55	1123.14	1100.91	322.51

Table 2: Comparison of models with Simulated data for N = 4500 and 10000

			N = 4500			
model	GBm	O-U	VAS	CEV	Merton	xMJNID
drift	0.21100	0.02577	0.67086	1.00004	0.21100	0.21101
diffusion	-0.48643	2.73895	2.73909	0.66879	0.48623	0.48627
mase	0.67436	0.19118			0.43119	0.43119
rmse	1.76449	0.52283	0.73909	1.18491	1.08075	1.08072
smape	2.00000	1.15593	0.31191	1.55721	1.60886	1.60884
AIC	-4574.2	-3039.9	-3043.5	-4457.3	-4567.9	-163463.7
BIC	-4561.4	-3027.1	-3024.3	-4438.1	-4535.8	-163425.2
			N = 10000			
drift	0.00998	0.05438	0.20996	1.00004	0.01005	0.01001
diffusion	0.32277	1.52214	1.52214	0.41958	0.32275	-0.32277
mase	0.42180	0.13306			0.42182	0.583194
rmse	1.18600	0.34008	0.47786	1.28292	1.18602	1.642459
smape	1.72208	1.13567	0.27134	1.47403	1.72210	2.00000
AIC	-25407.1	-18510.2	-18510.9	-24607.9	-25400.3	-809721
BIC	-25392.7	-18495.8	-18489.2	-24586.3	-25364.3	-809678

The comparison of the parameters of the models was done bearing in mind the different parameterisation of the models in defining the drift and diffusion coefficients, this means some drawbacks in the comparisons of the parameters' values, affecting the accuracy measures. However, we are comparing the models using the model diagnostics AIC and BIC. xMJNID model has the lowest MASE, SMAPE, AIC and BIC when N = 200, followed by Merton model and GBM. Similar patterns was observed for N = 500. For N = 4500 and 10000, O-U model seemed to favour the parameter values than the rest models, however, the xMJNID model still had the least AIC and BIC values, indicating that it was the preferred model to the rest.

3.2 Data description of the Nigeria Exchange rate to the US Dollar

We present the dataset of the Nigeria Exchange rate to the US Dollar for data analysis and application of our modified geometric Brownian motion with jump and stochastic volatility. The monthly exchange rate data of the Nigeria naira to the US dollars was from the period of January 2000 to December 2023.

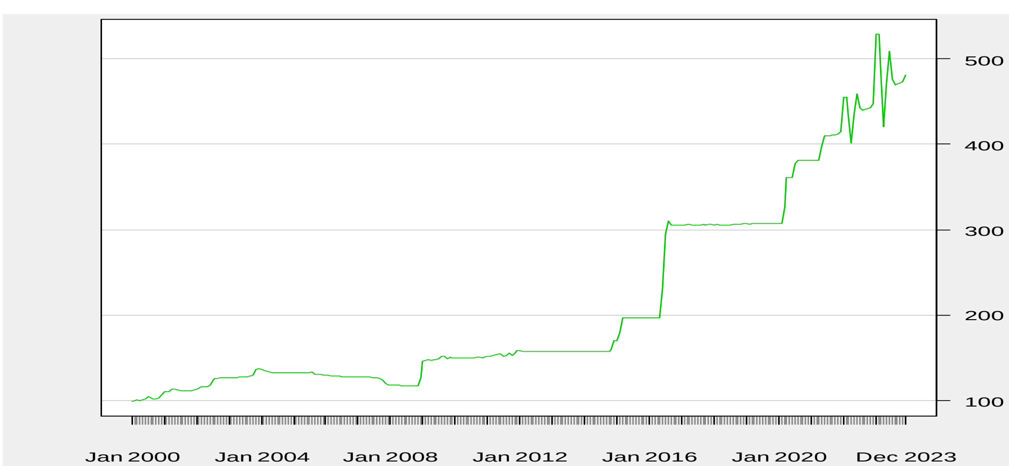

Figure 1. Time Series Plot of Nigeria Exchange Rate to US dollars

From the time series plots in Figure 1 above, there has been heightened variance in the exchange rate of the naira to US dollar, with the earlier periods being characterized by low prices and very high levels for the recent periods. Moreover, the price series is non-Gaussian as opposed to the log returns' series.

Autoregressive Integrated Moving Average model (ARIMA)

We applied the ARIMA model to the raw exchange rate of the naira to the US dollar by use of the *auto-arima* function in R and obtained as the best model ARIMA(2,1,0) with drift, AIC=2107.28, BIC=2121.92, log likelihood = -1049.64 and variance of 88.74. Applying the same function on the dataset with the monthly frequency in the function, we obtained a best model seasonal ARIMA(2,1,2)(1,0,0)[12] with AIC=2042.67, BIC=2064.63, log likelihood = -1015.34 and variance of 68.84. The ARIMA models are summarised in Table 3.

Table 3: Comparison of ARIMA models for Nigeria Exchange rate

	ARIMA(2,1,0) with drift	ARIMA(2,1,2)(1,0,0)[12]
AIC	2107.28	2042.67
BIC	2121.92	2064.63
Log likelihood	-1049.64	-1015.34
Variance	88.74	68.84

Comparing the two ARIMA models using the model comparison measures as shown in Table 3, shows that the seasonal ARIMA model is a better fit for the exchange rate data, as its measures are lower than that of the ARIMA with drift model. We can infer that the results may be because of the definition of the monthly frequency in the auto-arima function. Table 4 shows the model coefficients of the seasonal ARIMA and what follows is the derivation of the fitted seasonal ARIMA model for the exchange rate of the naira to the US dollar Where ε_t is white noise with standard deviation of $8.297=\sqrt{68.84}$.

Table 4: ARIMA(2,1,2)(1,0,0)[12] for Nigeria Exchange rate

Sigma sqd. = 68.84					
	ar1	ar2	ma1	ma2	sar1
Coeff.	-0.1362	-0.6594	0.4935	0.5555	0.6286
s.e.	0.1263	0.087	0.1441	0.1008	0.0753

The seasonal ARIMA with two autoregressive parts, two moving average parts and a seasonal autoregressive part is given thus

$$\hat{y}_t = 1.358y_{t-1} + 0.652y_{t-2} - 0.822y_{t-3} + 0.585y_{t-4} + 0.366y_{t-5} - 0.629\varepsilon_{t-12} + \varepsilon_t$$

where ε_t is white noise with standard deviation of $8.297 = \sqrt{68.84}$.

The plot of the exchange rate of the naira to the dollar from the seasonal ARIMA for 2024 and 2025 is shown in Figure 2. The forecast is the blue line, and the 80% and 95% prediction intervals are shown by the dark grey and light grey shaded area around the blue line, respectively.

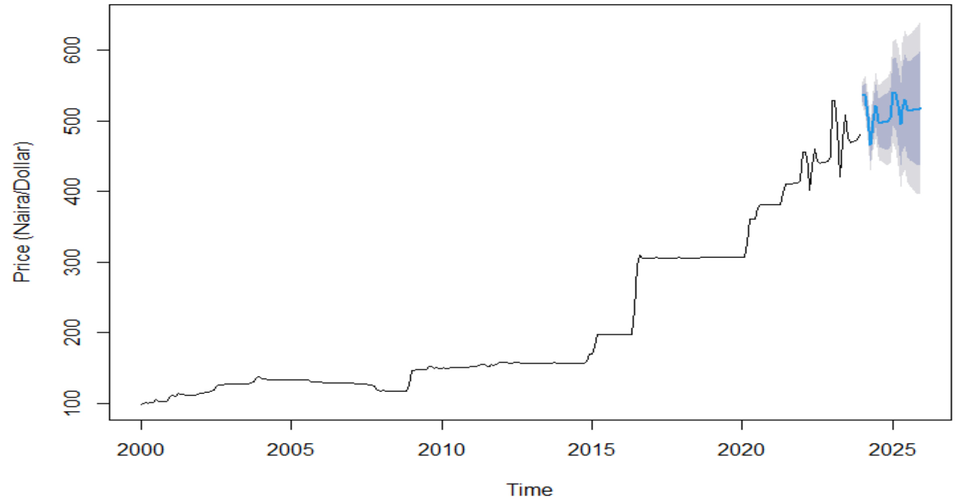

Figure 2. Plot of Exchange rate Forecast from ARIMA(2,1,2)(1,0,0)[12]

The realities of the exchange rate given all economic, political and social indicators for the year 2024 has shown that this forecast from the seasonal ARIMA had been further from the realities. And we particularly noted the absence of jumps in the forecast for which we saw was a key characteristic of the exchange rate of the naira to the US dollar.

3.3 Modelling Naira/Dollar exchange rate with xMJNID model

At this juncture, we shall investigate the performance of xMJNID model with various diffusion processes on the naira/dollar exchange rate using AIC, BIC and log-likelihood as well as their estimate standard errors. The diffusion processes models we used for comparison with our xMJNID model includes but not limited to the models discussed in chapter three, Geometric Brownian motion (GBm), Ornstein–Uhlenbeck (O-U), Vasicek Model (VAS), Constant Elasticity of Variance (CEV), and Merton model. The estimates, AIC and BIC are given in Table 5.

Table 5: Comparison of Diffusion Processes on Naira/Dollar exchange rate data

model	drift	std. err	diffusion	std. err	AIC	BIC	Log-like.
GBm	1.51331	0.48320	-0.51566	0.02209	1842.83	1850.15	1838.83
O-U	-1.45866	0.65901	167.95256	7.10160	2167.62	2174.94	2163.62
VAS	1.46454	na	0.00258	na	3456.97	3467.95	3450.97
CEV	1.56348	0.51745	1.58605	na	1892.36	1903.34	1886.36
Merton	0.55318	0.18166	0.18698	0.00878	1284.56	1302.86	1274.56
xMJNID	0.55321	0.18166	0.18699	0.00878	1247.26	1269.22	1235.26
ARIMA with drift					2107.28	2121.92	-1049.64
Seasonal ARIMA					2042.67	2064.63	-1015.34

* na = not available

As expected all the models are to give different estimates, however we can see that those models that are similar in their structure have close estimates for the drift coefficient, like the diffusion processes of Geometric Brownian motion (GBm), Ornstein–Uhlenbeck (O-U), Vasicek Model (VAS) and Constant Elasticity of Variance (CEV). Their diffusion coefficient estimates are very different except for VAS. We note that the estimates of our xMJNID model are close to that of Merton model with very similar standard error of the estimates. Looking at the AIC, BIC and log-likelihood, shows that xMJNID model has the lowest AIC and BIC with values closest to those of Merton model (xMJNID has AIC and BIC of 1247.26 and 1269.22, while Merton model had AIC and BIC of 1284.56 and 1302.86 respectively).

3.4 Dynamics of jumps in xMJNID model for the exchange rate

The jump component is composed of lognormal jumps driven by a Poisson process and models the sudden changes in the exchange rate due to the arrival of new important information, whether economic or political. The summary model statistics from our PM model is shown in Table 6a and Table 6b.

Table 6a: Summary statistics from xMJNID model on Naira/Dollar exchange rate data

Coef.	Estimate	Std. Error
diffusion	0.187	0.009
drift	0.553	0.182
alpha	2.964	17.093
lambda	18.781	18.647
beta	14.962	8.874
delta	39.651	7.034
Number of estimated jumps: 20		
Average inter-arrival times: 0.036		
Average jump size: 13.909		
Standard Dev. of jump size: 37.796127		
Jump Threshold: 15.87		

Notice that the summary statistics for jump size (Standard Dev. of jump size: 37.796127) suggest that the jumps come from the Gaussian distribution with zero mean and standard deviation 37.796127.

Table 6b: Summary statistics from xMJNID model on Naira/Dollar exchange rate data

Summary statistics for jump times:					
Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
0.4246	0.9593	1.0536	0.9786	1.0933	1.1151

Summary statistics for jump size:					
Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
-68.75	-17.06	19.48	13.91	35.34	81.41

The forecast of the paths with jumps from the xMJNID model is also shown in Figure 3, where the forecast values are circled in red text on the plot of the exchange rate data, indicating the jump process fitted by the xMJNID. The summary statistics of the jump times and size are given in Table 4b. We can observe from the results that the mean jump time is 0.9786 and the mean jump size is 13.91.

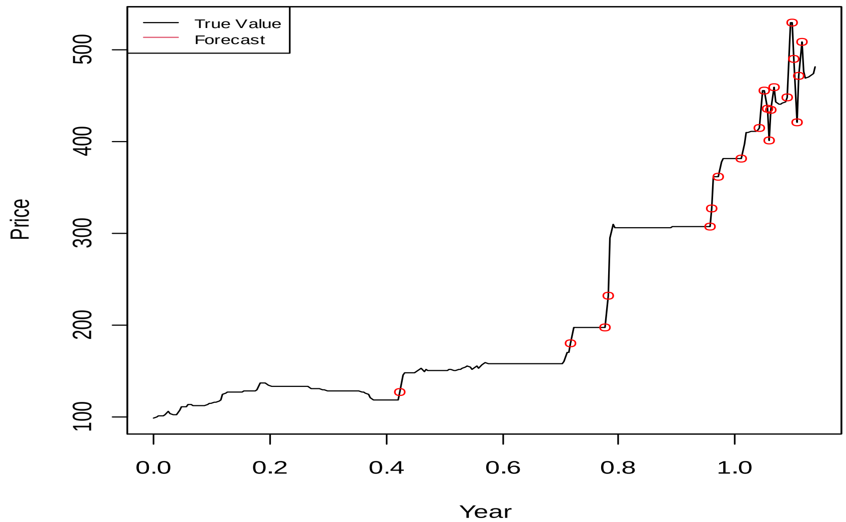

Figure 3. Plot of Exchange rate and xMJNID Model Forecast

Conclusion

The xMJNID estimates revealed a drift of 0.553, diffusion of 0.187, lambda of 18.781, beta of 14.962 and delta of 39.651. It had 20 estimated jumps with an average jump size of 13.909 and 15.87 jump threshold. Given the structure of xMJNID with a diffusion part and a jump component, there are very rear periods of normal price distribution (Gaussian) but more of random intermittent unusual price movements or jumps. For our dataset, the variance of the diffusion though positive was not significant at 0.187 but the jump component was both positive and significant at 18.781, implying that jumps were quite dominant in the naira/dollar exchange rate dynamics. Moreover, the intensity of the compound Poisson, i.e. the probability of the jump is high, because the drift term was significantly at 0.553 indicating a presence an upward trend in the naira/dollar exchange rate.

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