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OPTIMAL DEBT RATIO AND CONSUMPTION PLAN FOR AN INVESTOR IN THE PRESENCE OF INFLATION RISK AND CORPORATE TAX

C. B. Ibe and O.E. Daudu

Department of Mathematics, Faculty of Physical Sciences, University of Benin, P.M.B 1154, Benin City, Nigeria

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ABSTRACT

In this paper, we derive the optimal debt ratio and optimal consumption strategy for an investor in the presence of inflation risk and corporate taxation. The investor is assumed to pay tax on income generated by its asset. The investor's income is assumed to grow at rate that satisfy a diffusion process. The aim of the investor is to derive the real wealth process which is nominal wealth adjusted for inflation. The resulting real wealth process was solved using dynamic programming approach. As a result, we derive the optimal debt ratio and optimal consumption rate for the investor over time by assuming that the investor chooses a power utility function. We found that the debt ratio depends positively on the corporate tax rate, debt servicing and the volatility of the inflation index. Also, we found that as the risk aversion coefficient increases, the optimal debt ratio will decrease and vice versa. Also, we found that the relationship between the coefficient of risk aversion and the optimal consumption rate is positive.

1. Introduction

The optimal management of debt is an important aspect of corporate financing today. One of the reason for this is because companies now routinely rely on debt financing for expansion, asset acquisition and general operational support. However, while debt financing can offer some strategic advantage, it also increases the financial leverage of firms that can threaten the viability of the firm if not properly managed. The optimal debt ratio represents the proportion of a company's assets financed through debt and leveraging debt can amplify returns on investments.
Hence, having a good understanding of the factors that impact on a firm's debt ratio is necessary for a firm that wants to survive in the current complex financial world. The debt ratio gives an insight into the financial health of a firm and represent an important factor that may influence the investment and financing decisions of firms.

*Corresponding author: IBE, C.B.
E-mail address: chiedozie.ibe@uniben.edu
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Also, the performance of a firm's investment is affected by debt exposure especially in periods of a financial crisis and how well the firm is able to manage this exposure is critical for its survival. The business activities of firms in Nigeria are exposed to several risks which include: economic risk, political risk, inflation risk, regulatory risk, currency risk, corruption etc. In this paper, we consider inflation, investment and income growth risks and address the problem of optimal debt ratio and consumption strategy for the investor.

The literature is rich with works on optimal debt management. For example, [1] studied the optimal debt ratio and consumption plan for an investor during financial crisis in a stochastic setting. The impact of labor market condition was also studied. He assumed that the production rate function of the investor is stochastic and being influenced by government policy, employment and unanticipated risks. [2] considered a stochastic optimal control model and optimal debt ratio management strategies of an investor in a financial crisis. They considered productivity of capital, asset return, interest rate and market regime switches of an investor. The utility of terminal wealth was optimized under debt ratio. [3] considered backup security as a buffer for loans and derived the optimal portfolio, debt ratio and consumption rate for an investor during a financial crisis. [4] considered the surplus process of an insurer which satisfies a diffusion process and obtained an optimal policy for debt and dividend payment for an insurer. Further, while [5] considered the optimal debt ratio, investment and dividend payment strategies for an insurance company in a finite time horizon by maximizing the total expected discounted utility of dividend payment, [6] considered the optimal liability management strategies and dividend payment for an insurance company that experiences catastrophic risks. [7] considered the optimal investment strategies and collaterized optimal debt in the presence of intangible asset within a jump-diffusion framework. [8] provided a methodology to estimate the optimal debt ratio when asset returns follow a geometric Brownian motion but adjusted by the probability of default that follows an Ornstein-Uhlenbeck process.

We are also interested on the impact of inflation on the wealth of the investor and how it affects their consumption and debt management strategies. In fact, real wealth rather than nominal wealth serves as a guide to investors in deciding how much capacity they have for investment and consumption. [9] considered a stochastic inflation index with jumps while [10] used a stochastic dynamic programming approach to model a DC pension in a complete market. They considered the impact of both income and inflation risk on the model. For more reviews on inflation dynamics [11], [12] and [13].

In this paper, the investor we consider operates within a stochastic inflation framework. The asset price, inflation index and income growth rate dynamics are assumed to follow a diffusion process. The wealth process of the investor is described as the difference between asset value and debt. To account for the impact of inflation risk, we derived the real wealth process of the investor. We considered two state variables in our value function which include the real wealth process and income growth rate. The goal of the investor is to maximize the discounted expected utility of consumption over an infinite horizon. Using the stochastic dynamic programming techniques, we obtained the optimal debt ratio and optimal consumption strategy by assuming a Constant Relative Risk Aversion (CRRA) utility function.

The rest of the paper is organized as follows. A general formulation of the model is presented in section 2. In section 3, we presented the optimal control problem. Some numerical results for the models is presented in section 4. We conclude the paper in section 5.

2 The Model Formulation

Let $A(t)$ be the asset value of the firm at time t, $B(t)$ is the price of the firm's assets at time t and $K(t)$ is the quantity of assets owned by the firm at time t such that $A(t) = B(t)K(t)$.
We assume that the asset price $B(t)$ satisfies the following equation

$$\frac{dB(t)}{B(t)} = \mu_B(t)dt + \sigma_B(t)dW_B(t), \tag{1}$$

where $\mu_B(t)$ is the expected growth rate of asset price at time t, $\sigma_B(t)$ is the volatility of asset price at time t and $W_B(t)$ is a standard Brownian motion that captures sources of asset price risk at time t.
Due to the stochastic nature of the asset price of the firm, we can determine the dynamics of the firm's asset value. Hence, we have that the change in asset value of the firm satisfies the following dynamics

$$\begin{aligned} dA(t) &= B(t)dK(t) + dB(t)K(t) = B(t)dK(t) + A(t)\frac{dB(t)}{B(t)} \\ &= B(t)dK(t) + A(t)\left(\mu_B(t)dt + \sigma_B(t)dW_B(t)\right) \end{aligned} \tag{2}$$

Observe that change in $A(t)$ is caused by a combination of two components: a change in the asset price given by $A(t)(\mu_B(t)dt + \sigma_B(t)dW(t))$ and a change in the asset quantity $B(t)dK(t)$. Note that the change in asset quantity is considered an investment by the firm and thus will be captured as part of the firm's expenditure.
Let $X(t)$ *and* $L(t)$ be the wealth and liability of the investor respectively. The liability $L(t)$ at time t is described as the difference between the firm's expenditure and income. The components of $L(t)$ includes interest rate paid on debt r_L at time t, the amount consumed by the investor at time t given by $dC(t) = c(t)X(t)dt$, where $c(t)$ is the consumption rate and a change in asset quantity $B(t)dK(t)$.

Let $E(t)$ be the expenditure process of the firm at time t. It follows that $E(t)$ satisfies the following dynamics

$$dE(t) = r_L(t)L(t)dt + B(t)dK(t) + c(t)X(t)dt. \tag{3}$$

Next, we consider the income generated by the asset of the investor at time t. Let $Y(t)$ be the income generated by the assets of the investor at time t and we assume that it is taxable by the government. In other words, the firm pays corporation tax to government on the income generated by its assets. Let $\eta(t)$ be the income growth rate of the firms asset at time t. It then follows that

$$dY(t) = \eta(t)(1-\tau)A(t)dt, \tag{4}$$

where $0 \le \tau < 1$ is the corporation tax.

The income growth rate in this study is assumed to be stochastic in nature as the assumption of a constant income growth especially over a long period of time might not be realistic. Following Jin (2014), $\eta(t)$ is assumed to evolve according to the following diffusion process

$$d\eta(t) = \left[\beta\big(\eta(t)\big) + \eta(t)\phi(\omega)\right]dt + \sigma_\eta(t)dW_\eta(t),\ \eta(0) = \eta_0, \tag{5}$$

where $\beta\big(\eta(t)\big): \mathbb{R} \times [0,T] \to \mathbb{R}$ is the expected growth rate of $\eta(t)$, $\phi(\omega): \mathbb{R} \to \mathbb{R}$ represents the impact of productive capacity on the firm's asset. $\sigma_\eta(t)$ is the volatility of the income growth at time t and $W_\eta(t)$ is a standard Brownian motion that captures sources of income growth rate risk at time t. The net change in liability is given by

$$dL(t) = r_L(t)L(t)dt + B(t)dK(t) + c(t)X(t)dt - \eta(t)(1-\tau)A(t)dt \qquad (6)$$

Proposition 1. *The wealth process of the investor at time t is*

$$\frac{dX(t)}{X(t)} = \left[\left(\mu_B(t) + \eta(t)(1-\tau)\right)\left(1+\varphi(t)\right) - r_L(t)\varphi(t) - c(t)\right]dt + (1+\ \varphi(t))\sigma_B(t)dW_B(t), \ \ X(0) = x_0.$$

Proof. By definition, we have that the wealth process is given by

$$X(t) = A(t) - L(t) \qquad (7)$$

Taking the differential of both sides of (7), we have

$$dX(t) = dA(t) - dL(t)$$

Using (2) *and* (6), we have that

$$dX(t) = B(t)dK(t) + A(t)[\mu_B(t)dt + \sigma_B(t)dW_B(t)] - r_L(t)L(t)dt - B(t)dK(t) - c(t)X(t)dt + \eta(t)(1-\tau)A(t)dt \qquad (8)$$

But $A(t) = X(t) + L(t)$, which implies that

$$dX = (X(t) + L(t))[\mu_B(t)dt + \sigma_B(t)dW_B(t)] - r_L(t)L(t)dt - c(t)X(t)dt + \eta(t)(1-\tau)(X(t)+L(t))dt \qquad (9)$$

Hence, the result

$$\frac{dX(t)}{X(t)} = \left[\left(\mu_B(t) + \eta(t)(1-\tau)\right)\left(1+\varphi(t)\right) - r_L(t)\varphi(t) - c(t)\right]dt + (1+\ \varphi(t))\sigma_B(t)dW_B(t), \qquad (10)$$

where $\varphi(t) = \frac{L(t)}{X(t)}$ is the liability ratio.

Now, we adjust the nominal wealth of the investor for inflation to obtain the real wealth. But first, we consider the dynamics of the inflation index. The consumer price index is a measure that helps us appreciate the real impact of price changes on goods and services. The prices of goods and services in Nigeria for example have experienced significant fluctuations in recent times. Hence, in this paper, the inflation index is assumed to evolve according to the following diffusion process

$$\frac{dI(t)}{I(t)} = \mu_I(t)dt + \sigma_I(t)dW_I(t) \quad I(0) = I_0 > 0, \qquad (11)$$

where $\mu_I(t) = r(t) - \bar{r}(t) + \sigma_I(t)\theta_I(t)$ is the expected inflation rate at time t, $r(t)$ is the nominal interest rate at time t, $\bar{r}(t)$ is the real rate of interest at time t, $\sigma_I(t)$ is the volatility of the inflation index at time t, $W_I(t)$ is a standard Brownian motion that captures sources of inflation risk at time t and $\theta_I(t)$ is the market price of inflation risk at time t.

Definition 1. *The real wealth of the investor is nominal wealth adjusted for inflation and it is given by* $\bar{X}(t) = \frac{X(t)}{I(t)}$ (12)

We shall no longer indicate the functional dependencies unless it becomes necessary to do so.

Proposition 2.

$$d(\bar{X}) = d\left(\frac{X}{I}\right) = \bar{X}\left[\left(\mu_B + \eta(1-\tau)\right)(1+\varphi) - r_L\varphi - c - \mu_I + \sigma_I^2 - \rho(1+\varphi)\sigma_B\sigma_I\right]dt + \bar{X}(1+\varphi)\sigma_B dW_B - \bar{X}\sigma_I dW_I, \ \ \bar{X}(0) = \bar{x} > 0 \qquad (13)$$

Proof. Taking the differential of both sides of (12), we have

$$d(\bar{X}) = d\left(\frac{X}{I}\right) \tag{14}$$

Applying the Ito quotient rule for stochastic differential equations on (14), where $X \; and \; I$ satisfies (10) and (11) respectively, the result follows immediately.

3. The optimal control problem

In this section, the admissible control strategies, optimal controls and the value function for our problem is presented.

We now operate in a filtered probability space $(\Omega, F_t, \{F_t\}, \mathbf{P})$, where F_t is the σ-algbera generated by $\{W_I(t), W_B(t), W_\eta(t): 0 \le s \le t\}$, $\{F_t\}$ is the filtration and **P** is the real world probability.

Definition 2. *The strategies* $(\cdot) = \{\varphi, c: t \ge 0\}$ *that is progressively measurable with respect to* $\{W_I(t), W_B(t), W_\eta(t): 0 \le s \le t\}$ *is called an admissible strategy.*

For admissible debt ratio φ, *we assume that for all* $T \in (0, \infty)$, $E\int_0^T \varphi^2 \, dt < \infty$,

For admissible consumption rate c, *we assume that for all* $T \in (0, \infty)$, $E\int_0^T c^2 \, dt < \infty$,

Let $\boldsymbol{A}$ *be the collection of all admissible strategies, then the collection of admissible controls is defined as*

$$\boldsymbol{A} = \left\{u = (\varphi, c: t \ge 0) \in \mathbb{R} \times \mathbb{R}: E\int_0^T \varphi^2 \, dt < \infty, E\int_0^T c^2 \, dt < \infty\right\}. \tag{15}$$

The desire of the investor is to choose a liability ratio and consumption rate, that will optimize the expected discounted utility of consumption in an infinite time horizon. For an arbitrary admissible strategy $(\cdot) = \{\varphi, c: t \ge 0\}$, the objective function is given as

$$J(\bar{x}, \eta; u) = \int_t^\infty \left(e^{-\delta s} H\big(c(s)\big)\right) ds, \tag{16}$$

where $H(c(s))$ is a utility function with respect to consumption and $0 \le \delta < 1$ is the discount rate.

We define the value function

$$G\big(t, \bar{X}(t), \eta(t)\big) := \sup_{u \epsilon A} \left(J(\bar{x}, \eta; u) | \bar{X}(t) = \bar{x}, \eta(t) = \eta\right) \tag{17}$$

Applying the stochastic dynamic programming approach, the Hamilton-Jacobi-Bellman (HJB) equation that characterizes the optimal solutions to the problem of the firm becomes

$$\begin{aligned} 0 = G_t &+ \left[\big(\mu_B + \eta(1-\tau)\big)(1+\varphi) - r_L\varphi - c - \mu_I + \sigma_I^2 - \rho(1+\varphi)\sigma_B\sigma_I\right]\bar{x}G_{\bar{x}} \\ &+ \left[\beta\big(\eta(t)\big) + \eta(t)\phi(\omega)\right]G_\eta + \frac{1}{2}\bar{x}^2(1+\varphi)^2\sigma_B^2 G_{\bar{x}\bar{x}} + \frac{1}{2}\bar{x}^2\sigma_I^2 G_{\bar{x}\bar{x}} + \frac{1}{2}\sigma_\eta^2 G_{\eta\eta} \\ &+ \bar{x}(1+\varphi)\rho_2\sigma_B\sigma_\eta G_{\eta\bar{x}} - \bar{x}\sigma_B\sigma_\eta\rho_3 G_{\eta\bar{x}} + e^{-\delta}H(c) \end{aligned} \tag{18}$$

with the transversality condition $\lim_{t\to\infty} E[G(t, \bar{x}, \eta)] = 0$, $G_t = \frac{\partial G}{\partial t}$, $G_{\bar{x}} = \frac{\partial G}{\partial \bar{x}}$, $G_\eta = \frac{\partial G}{\partial \eta}$, $G_{\bar{x}\bar{x}} = \frac{\partial^2 G}{\partial \bar{x}^2}$, $G_{\eta\eta} = \frac{\partial^2 G}{\partial \bar{\eta}^2}$, $G_{\bar{x}\eta} = \frac{\partial^2 G}{\partial \eta \partial \bar{x}}$.

By the standard homogeneity argument for infinite-horizon problems, we have that

$$e^{\delta t} G(t, \bar{x}, \eta) = \sup_{\{\varphi(s), c(s): t \le s \le \infty\}} E_t \int_t^\infty [e^{-\delta(s-t)} H\big(c(s)\big)] ds$$

$$= \sup_{\{\varphi(t+u),c(t+u):0\le u\le\infty\}} E_t \int_t^\infty [e^{-\delta(u)} H(c(t+u))] d$$

$$= \sup_{\{\varphi(u),c(u):0\le u\le\infty\}} E_0 \int_0^\infty [e^{-\delta(u)} H(c(u))] du$$

$$\equiv V(t,\bar{X},\eta) \tag{19}$$

which is independent of time. The third equality in this argument makes use of the fact that the optimal controls is Markov. Hence, $G(t,\bar{X},\eta) = e^{\delta t} V(t,\bar{x},\eta)$ and (18) reduces to the following time-homogeneous value function V:

$$\begin{aligned} 0 = V_t &+ \left[\left(\mu_B + \eta(1-\tau)\right)(1+\varphi) - r_L\varphi - c - \mu_I + \sigma_I^2 - \rho(1+\varphi)\sigma_B\sigma_I\right]\bar{x}V_{\bar{x}} \\ &+ \left[\beta\left(\eta(t)\right) + \eta(t)\phi(\omega)\right]V_\eta + \frac{1}{2}\bar{x}^2(1+\varphi)^2\sigma_B^2 V_{\bar{x}\bar{x}} + \frac{1}{2}\bar{x}^2\sigma_I^2 V_{\bar{x}\bar{x}} + \frac{1}{2}\sigma_\eta^2 V_{\eta\eta} \\ &+ \bar{x}(1+\varphi)\rho_2\sigma_B\sigma_\eta V_{\eta\bar{x}} - \bar{x}\sigma_B\sigma_\eta\rho_3 V_{\eta\bar{x}} + e^{-\delta}H(c) \end{aligned} \tag{20}$$

with transversality condition $\lim_{t\to\infty} E[V(t,\bar{x},\eta)] = 0$.

The investor has the following power utility function $G(c) = \frac{C^{1-\gamma}}{1-\gamma}$ with respect to consumption for $\gamma > 0$, where γ is the constant relative risk aversion parameter.

We assume a solution to (20) of the form

$$V(t;\bar{x},\eta) = \frac{\bar{x}^{1-\gamma}}{1-\gamma} e^{h(t,\eta)} \tag{21}$$

From (21), we obtain the following partial derivatives

$$V_{\bar{x}} = \bar{x}^{(-1)}(1-\gamma)V(t;\bar{x},\eta), \qquad V_{\bar{x}\bar{x}} = -\bar{x}^{(-2)}\gamma(1-\gamma)V(t;\bar{x},\eta)$$
$$V_\eta = h_\eta V(t;\bar{x},\eta), \qquad V_{\eta\eta} = \left(h_{\eta\eta} + h_\eta^2\right)V(t;\bar{x},\eta)$$
$$V_{\eta\bar{x}} = \bar{x}^{(-1)} h_\eta (1-\gamma)V(t;\bar{x},\eta), \qquad V_t = h_t V(t;\bar{x},\eta) \tag{22}$$

Substituting (22) into (20) and dividing through by $(1-\gamma)V(t;\bar{x},\eta)$, we have

$$\begin{aligned} 0 = \frac{h_t}{(1-\gamma)} &+ \left[\left(\mu_B + \eta(1-\tau)\right)(1+\varphi) - r_L\varphi - c - \mu_I + \sigma_I^2 - \rho(1+\varphi)\sigma_B\sigma_I\right] \\ &+ \left[\beta\left(\eta(t)\right) + \eta(t)\phi(\omega)\right]\frac{h_\eta}{(1-\gamma)} - \frac{1}{2}\gamma(1+\varphi)^2\sigma_B^2 + \frac{1}{2(1-\gamma)}\sigma_\eta\sigma'_\eta\left(h_{\eta\eta} + h_\eta^2\right) \\ &- \frac{1}{2}\gamma\sigma_I^2 + (1+\varphi)\rho_2\sigma_B\sigma_\eta h_\eta - \sigma_B\sigma_\eta\rho_3 h_\eta + \frac{C^{1-\gamma}}{1-\gamma}\bar{x}^{-(1-\gamma)}e^{-h(t,\eta)} \\ &- \frac{\delta}{(1-\gamma)} \end{aligned} \tag{23}$$

From (23), we obtain the optimal liability ratio of the investor as:

$$\begin{aligned} \varphi^* = 1 &+ \left(r_L - \mu_B\right)(\gamma\sigma_B^2)^{-1} - \eta(1-\tau)(\gamma\sigma_B^2)^{-1} + \rho\sigma_I(\gamma\sigma_B)^{-1} \\ &- \sigma_\eta\left(1 + \rho_2 h_\eta\right)(\gamma\sigma_B)^{-1} \end{aligned} \tag{24}$$

The equation shows that the investor's optimal liability is decreasing in the volatility of asset price and the coefficient of risk aversion. In fact, the optimal levels of liability indicate that before an investor makes a decision concerning the level of liability to incur to increase its portfolio, it has to consider the amount to be spent on debt servicing, the tax adjusted income growth rate, the impact of inflation risk, asset price volatility and the coefficient of risk aversion.

Now, we examine the reaction of the optimal debt ratio φ^* to a change in the corporate tax rate τ. The derivative is

$$\frac{\partial \varphi^*}{\partial \tau} = \eta(\gamma\sigma_B^2)^{-1} > 0. \tag{25}$$

The optimal debt ratio depends positively on the corporate tax rate. This result is supported by Graham et al. (1998) which found that there is a positive relationship between tax rates and debt levels. Observe that the extent to which a change in the corporate tax rate impacts positively on the debt ratio depends on the income growth rate η, the diffusion risk of asset price and the CRRA coefficient γ. Further, $\lim_{\gamma\to\infty} \frac{\partial \varphi^*}{\partial \tau} = 0$, indicates that with a change in τ, a highly risk averse investor will be unwilling to incur any debt with a changing corporate tax regime. However, $\lim_{\gamma\to 0} \frac{\partial \varphi^*}{\partial \tau} = \infty$ implies that with a change in τ, a risk loving investor will continue to look for debt financing opportunities that will help increase asset value even at the expense of an increasing debt ratio. This result is very significant as it indicates that even with a changing corporate tax regime, the risk loving investor will increase liability if the right financing opportunity presents itself.

The impact of a change in the liability ratio given a change in the corporate tax rate also depends on whether the firm's income growth rate is positive or negative. We observe that if $\eta > 0$, then $\frac{\partial \varphi^*}{\partial \tau} > 0$ and if $\eta < 0$, then $\frac{\partial \varphi^*}{\partial \tau} < 0$.

We now examine the reaction of the optimal debt ratio φ^* to a change in the income growth rate η. We have that

$$\frac{\partial \varphi^*}{\partial \eta} = -(1-\tau)(\gamma\sigma_B^2)^{-1} < 0 \tag{26}$$

Observe that the change in liability with respect to income growth rate is negative. This implies that an investor with an increasing η will likely have more funds available to service debt and hence lead to a lower debt ratio. However, the extent of the impact of a change in η on φ^* depends on the corporate tax rate, CRRA coefficient and diffusion risk of asset price.

Also,

$$\frac{\partial \varphi^*}{\partial r_L} = (\gamma\sigma_B^2)^{-1} > 0 \tag{27}$$

Clearly, debt servicing is positively related to debt ratio. It simply means that an increase in the debt ratio will engender an increase in the amount spent by the investor on debt servicing. This result follows intuition.

$$\frac{\partial \varphi^*}{\partial \sigma_I} = \rho(\gamma\sigma_B)^{-1} > 0 \tag{28}$$

The impact of a change in the volatility of the inflation index on the optimal liability ratio is captured in equation (28). We observe that the relationship between inflation index and optimal liability is positive.

Next, we determine the optimal consumption plan of the investor. From (23), we have the function $f(c)$, where

$$f(c) = -c + \frac{C^{1-\gamma}}{1-\gamma}\bar{x}^{-(1-\gamma)}e^{-h(\eta)}. \tag{29}$$

Proposition 4. The optimal consumption rate c^* is given as

$$c^*(t) = \left(\bar{x}^{(1-\gamma)}\exp\big(h(\eta)\big)\right)^{-\frac{1}{\gamma}}.$$

Proof. By first order principle, we have that

$$\frac{\partial f(c)}{\partial c} = -1 + c^{\gamma-1}\bar{x}^{-(1-\gamma)}e^{-h(\eta)} = 0.$$

This implies that

$$c^*(t) = \left(\bar{x}^{(1-\gamma)} \exp\left(h(\eta)\right)\right)^{-\frac{1}{\gamma}}. \quad (30)$$

Clearly, the more risk averse an investor is i.e as the coefficient of risk aversion becomes increasingly large, consumption tends to unity. In other words, a risk averse investor is more likely to consume rather than seek for deficit financing that can help improve on his or her business.

4 Some Numerical Example

In this section, we present the numerical analysis of the model developed in this study. The base values of our parameter are given as follows $\gamma = 5$, $\tau = 0.17$, $r_L = 0.18$, $\sigma_I = 0.2$, $\eta = 0.15$, $\mu_B = 0.06$, $\sigma_B = 0.28$, $\sigma_\eta = 0.08$, $h_\eta = 0.15$, $\rho_2 = 0.13$ and $\rho = 0.2$. The following table values were generated by the use of Matlab software.

Table 1 show the change in the optimal liability ratio of the investor with respect to the interest rate on debt, tax rate, income growth rate and the coefficient relative risk aversion. It is observed that an increase in the amount paid by the investor as debt service charge will lead to an increase in the liability ratio. In other words, as the percentage of the investor's wealth that is invested in paying the interest rate on debt increases, the overall liability of the firm increases as well. This indicates that firms must make effort to ensure that the amount spent on servicing loans is at a manageable level.

Table 1: The change in φ^* with respect to r_L, τ and η.

r_L	φ^*	τ	φ^*	η	φ^*
0.03	0.5648	0.05	0.9015	0.02	1.2227
0.05	0.6158	0.08	0.9130	0.05	1.1591
0.10	0.7433	0.10	0.9206	0.07	1.1168
0.15	0.8706	0.12	0.9283	0.10	1.0533
0.18	0.9474	0.15	0.9398	0.12	1.0109
0.20	0.9984	0.17	0.9494	0.14	0.9686
0.25	1.1260	0.20	0.9589	0.18	0.8839
0.30	1.2535	0.25	0.9780	0.20	0.8415
0.35	1.3811	0.30	0.9972	0.25	0.7357
0.40	1.5086	0.40	1.0354	0.30	0.6298

It is also observed that an increasing tax rate will lead to an increase in the liability ratio of the firm. Clearly, an increase in government effort or drive to increase revenue through increase in the tax rate will have a negative impact on the overall liability of the firm and a greater exposure to the risk of defaulting on loans. Also from table 1, we observe that the increase in the income growth rate of the firm will lead to a reduction in the optimal liability ratio. It then implies that a firm that wants to reduce its overall liability must put in places policies that will engender a growth in income. Also in table 1, we observe that with an increasing coefficient of relative risk aversion, there is a decrease in the optimal liability ratio. This simply indicates that an investor that is risk averse is less likely to incur more debt that will likely increase its liability ratio. Clearly, risk loving investors are more likely to look for sources of capital even at the expense of a higher liability ratio.

Figure 1 shows the optimal debt ratio for varying values of the coefficient of relative risk aversion over time for all other parameters remain fixed. We observe that as the investor sentiment for taking risk reduces, the optimal liability decreases as well. In other words, there is an inverse relationship between the coefficient of risk aversion of the investor and the optimal liability ratio. This indicates that a risk averse investor is likely to have a lower debt ratio as they might not be willing to take advantage of available debt financing opportunities to increase portfolio return at the expense of a higher debt ratio.

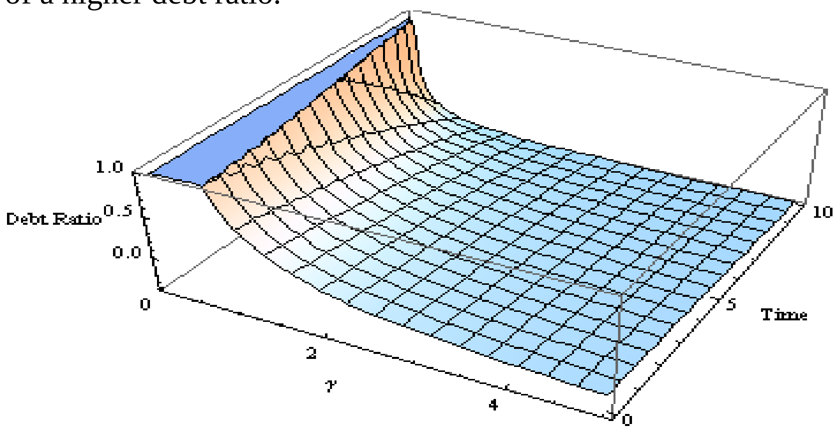

Figure 1: Optimal debt ratio for $r_L = 0.18,\ \eta = 0.15, \tau = 0.17,\ \sigma_\eta = 0.08,$
$\sigma_B = 0.28, \gamma = 5, \rho = 0.2\ and\ \mu_B = 0.06.$

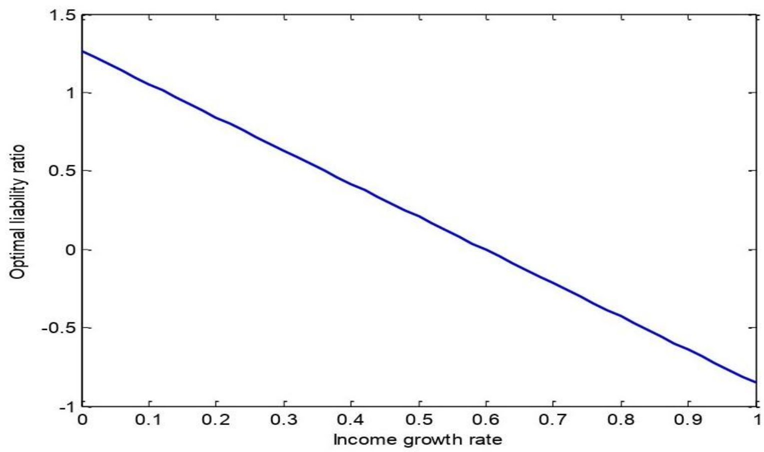

Figure 2: Optimal debt ratio vs income growth rate for $r_L = 0.18,\ \ \tau = 0.17,\ \sigma_\eta = 0.08,$
$\sigma_B = 0.28, \gamma = 5, \rho = 0.2\ and\ \mu_B = 0.06.$

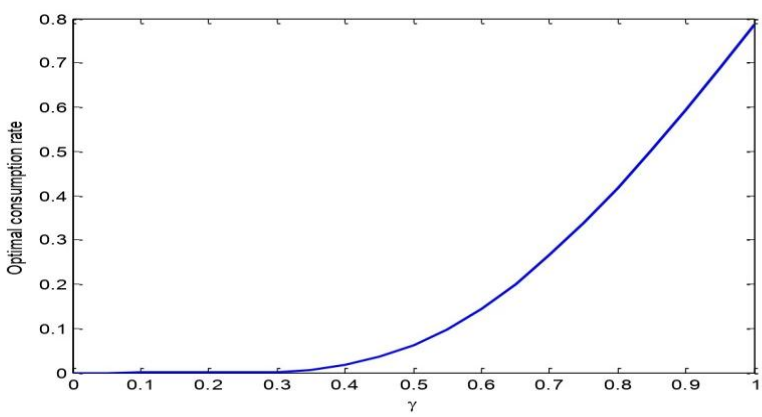

Figure 3: Optimal consumption rate vs coefficient of risk aversion for $x = 10$ and $h(\eta) = 0.24$.

Figure 2 shows the plot of optimal liability ratio vs income growth rate. We observe that there is an inverse relationship between income growth rate and debt ratio. In other words, an increase in the growth rate of the investor's income will lead to a decrease in the optimal liability ratio. Therefore, to have a lower debt ratio, policies that will help in increasing income must be adopted. Figure 3 shows the plot of optimal consumption rate vs the coefficient of risk aversion. Clearly, we see that an increase in γ leads to an increase in the consumption rate of the investor. In other words, a risk averse investor is more likely to consume than a risk loving investor.

Conclusion

In conclusion, we derived the optimal debt ratio and optimal consumption strategy of an investor that operates in a market with diffusion risks to maximize the expected discounted utility of consumption in an infinite time horizon. We found that the debt ratio depends positively on the corporate tax rate, debt servicing and the volatility of the inflation index.

From the numerical result, we found that:

- the consumption rate has a direct relationship with the coefficient of risk aversion.
- as the investor's coefficient of risk aversion increases, the debt ratio decreases and vice versa, suggesting that a risk averse investor is more likely to have a lower debt ratio.

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