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TROPICAL GEOMETRIC APPROACH TO SIGNED FULL TRANSFORMATION SEMIGROUPS

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ABSTRACT

Let $\Omega_k = \{1, 2,3, --, k\}$ be a finite set and ST_k be the signed full transformation semigroups. This work surveys the connection between tropical geometry and semigroup theory. Tropical geometry were applied on ST_k by degenerating elements from classical into tropical algebra. Through this approach, we analyze the structure of these transformations and determine the heights of their multiplicities through tropical curves. This study provides new perceptions into the algebraic properties of ST_k.

1. Introduction

A semigroup is an algebraic structure consisting of a non-empty set S together with an associative binary operation. A transformation of a set Ω_k is a function from Ω_k to itself. The transformation semigroup is one of the most fundamental mathematical objects that appear in theoretical computer science, where the properties of formal languages often depend on the algebraic properties of associated transformation semigroups.

Let $\Omega_k = \{1, 2,3, --, k\}$ be a finite set ordered in the standard way. A function $\alpha: Dom(\alpha) \subseteq \Omega_k \to Im(\alpha) \subseteq \Omega_k$ is said to be full (or total) if its image satisfies $Im(\alpha) = \Omega_k$, and it is denoted by T_k.

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A signed full transformation semigroup denoted by ST_k can be defined as the mapping $\alpha: Dom(\alpha) \subseteq \Omega_k \to Im(\alpha) \subseteq Z_k$ where $\Omega_k = \{1, 2, 3, \ldots, k\}$ and $Z_k = \{\pm 1, \pm 2, \pm 3, \ldots, \pm k\}$.

The term "tropical" was coined by French mathematicians to describe a new algebraic framework based on min-plus or max-plus arithmetic. Just as in classical algebra, a tropical polynomial $H(x) = \sum_{i=0}^{k} a_i\, x^i$ induces a tropical polynomial function denoted by H on T;

$$H:\ \mathrm{T} \longrightarrow \mathrm{T} \tag{1}$$

Tropical geometry reveals an unusual mathematical space with mysterious properties. Despite its unconventional structure, it satisfies fundamental geometric properties, making it a useful tool for various mathematical disciplines.

The set of tropical numbers is defined as $\mathrm{T} = \mathbb{R} \cup \{-\infty\}$ endowed with the operations called tropical addition and multiplication:

$$\text{"}\mu \oplus \nu\text{"} = \max\{\mu, \nu\} \tag{2}$$

$$\text{"}\mu \otimes \nu\text{"} = \mu \oplus \nu \tag{3}$$

with the usual conventions $\forall \mu \in \mathrm{T}$, "$\mu \oplus (-\infty)$"$=\max(\mu, -\infty) = \mu$ and

$$\text{"}\mu \otimes (-\infty)\text{"} = \mu \oplus (-\infty) = -\infty \tag{4}$$

Unlike classical arithmetic, tropical addition lacks an additive inverse, meaning tropical numbers form a semi-field rather than a full field. Notably, in tropical operations $2\mu \neq$ "$\mu \oplus \mu$" but "2μ" $= \mu \oplus 2$ and "0μ" $= \mu$ but not equal to 0. For example, $3 \oplus 5 = 5$ and $3 \otimes 5 = 8$.

Despite the growing interest in tropical geometry see [[2], [3], [4], [7], [8], [9], [10]], its applications in algebraic structures and semigroup theory see [[5], [11]], there has been limited investigation into its application to transformation semigroups particularly Signed Full Transformation Semigroups(ST_k). However, [12] examined the tropicalization of idempotent elements in full transformation semigroups. Recent findings on tropical geometry and transformation semigroups can be found in [[1], [6]]. This work connect tropical geometry with semigroup theory, specifically through the study of tropical polynomials and their multiplicities within the semigroup of ST_k. The elements in ST_k are transformed into tropical polynomials. Moreso, the study utilizes tropical curves to examine the structure of these transformations and determine their heights and multiplicities. Mathematical results including theorem and proposition were formulated to describe the properties of tropicalized elements in ST_k .Relevant examples are provided to validate the results.

Methods

The study begins by extracting the elements of the ST_k from T_k detailing its structure and properties. Elements of ST_k are then converted into tropical polynomials using tropical algebraic operations, specifically tropical addition and multiplication. The study examines these tropical polynomials and computes the multiplicities of their roots using GeoGebra software to illustrate them graphically. Theorem and proposition are formulated to describe the properties of the tropicalized elements, with proofs provided to support these results. Furthermore, tropical curves are analyzed to observe the structure of the transformations, and examples are presented to illustrate and confirm the theoretical findings.

Definition (Brugalle et. al. 2015)

A tropical number r is a root of a Polynomial $H(x)$ in one variable in which the points x_0 on the graph $H(x)$ has a corner at x_0 for $-\infty \le r \le \infty$.

Definition (Brugalle et. al. 2015)
The Multiplicity of a tropical root r denoted by $M(r)$ defined as
$M(r_n) = \sum_{i=1}^{n} |m_i - m_{i+1}|$ where m_i are the slopes of the lines in the tropical curve of $H(x)$ intersecting above r.

Proposition (Brugalle et al. 2015)
The tropical semi-field is algebraically closed. That is, every tropical polynomial of degree $d > 0$ has exactly d roots when counted with multiplicities.

Main Results

Theorem 1: Let $S = ST_k$. Then, every elements in ST_k has multiplicity of the same height for each n.
Proof: Let $S = ST_k$ and $\alpha, \beta \epsilon S$ such that $\alpha = \begin{pmatrix} \mu_1 & \mu_2 & \mu_3 - - - \mu_n \\ -\nu_1 & \nu_2 & -\nu_3 - - - \nu_n \end{pmatrix}$
and $\beta = \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 - - - \sigma_n \\ -\tau_1 & -\tau_2 & \tau_3 - - - \tau_n \end{pmatrix}$ for $a_n = c_n \in dom(\alpha, \beta)$ and $b_n \neq d_n \in Im(\alpha, \beta)$.

Consider the polynomials defined from $\alpha \; and \; \beta$ i.e

$$\alpha(x) = \mu_i x^n + \mu_{i+1} x^{n-1} + \mu_{i+2} x^{n-2} + \cdots + \mu_{i+k} x^{n-k} - \nu_i x^{n-1} + \nu_{i+1} x^{n-2} - \cdots + \nu_{i+k} x^{n-k}$$
and
$$\beta(x) = \sigma_i x^n + \sigma_{i+1} x^{n-1} + \sigma_{i+2} x^{n-2} + \cdots + \sigma_{i+k} x^{n-k} - \tau_i x^{n-1} - \tau_{i+1} x^{n-2} + \cdots + \tau_{i+k} x^{n-k}$$

having the tropical polynomials;

$$\mathsf{T}\,\alpha(x) = "\mu_i x^n + \mu_{i+1} x^{n-1} + \mu_{i+2} x^{n-2} + \cdots + \mu_{i+k} x^{n-k} - \nu_i x^{n-1} + \nu_{i+1} x^{n-2} - \cdots + \nu_{i+k} x^{n-k}"$$

$$\mathsf{T}\,\beta(x) = "\sigma_i x^n + \sigma_{i+1} x^{n-1} + \sigma_{i+2} x^{n-2} + \cdots + \sigma_{i+k} x^{n-k} - \tau_i x^{n-1} - \tau_{i+1} x^{n-2} + \cdots + \tau_{i+k} x^{n-k}"$$

Which gives;

$$max\{\mu_i + nx, \mu_{i+1} + (n-1)x, \ldots, \mu_{i+k} + (n-k)x, \nu_i - (n-1)x, \nu_{i+1} + (n-2)x, \nu_{i+k} + (n-k)x\}$$
$$max\{\sigma_i + nx, \sigma_{i+1} + (n-1)x, \ldots, \sigma_{i+k} + (n-k)x, \tau_i - (n-1)x, \tau_{i+1} - (n-2)x, \tau_{i+k} + (n-k)x\}.$$

Observing the tropical curve reveals that, α and β have distinct roots but share multiplicities of the same height. i.e, $|MT_{\alpha(x)}| = |MT_{\beta(x)}|$

■

Example 1: Let $\alpha, \beta \; \epsilon \; ST_{10}$, then consider the transformations $\alpha = (1-2), (2 \;\; 3 - 3], (4)(6 \;\; 5 - 4](7-5](8)(10 \;\; 9-9]$ and $\beta = (1-2), (2-3](\;3-3], (4)(5-4](6-5](7-5](8)(10 \;\; 9-9]$ with polynomial functions;

$$\alpha(x) = x^{10} + 6x^8 + x^7 + 9x^6 + 2x^5 + 12x^4 + 3x^3 + 17x^2 + x + 9$$

$$\beta(x) = x^{10} + x^7 + 9x^6 + 2x^5 + 2x^4 + 3x^3 + 17x^2 + x + 9$$

By the virtue of (2) and (3) we have;

$$\mathrm{T}\alpha(x) = max\{10x, 8x + 6, 7x, 6x + 9, 5x + 2, 4x + 12, 3x + 3, 2x + 17, x, 9\}$$

$$\mathrm{T}\beta(x) = max\{10x, 7x, 6x + 9, 5x + 2, 4x + 2, 3x + 3, 2x + 17, x, 9\}$$

Graphically,

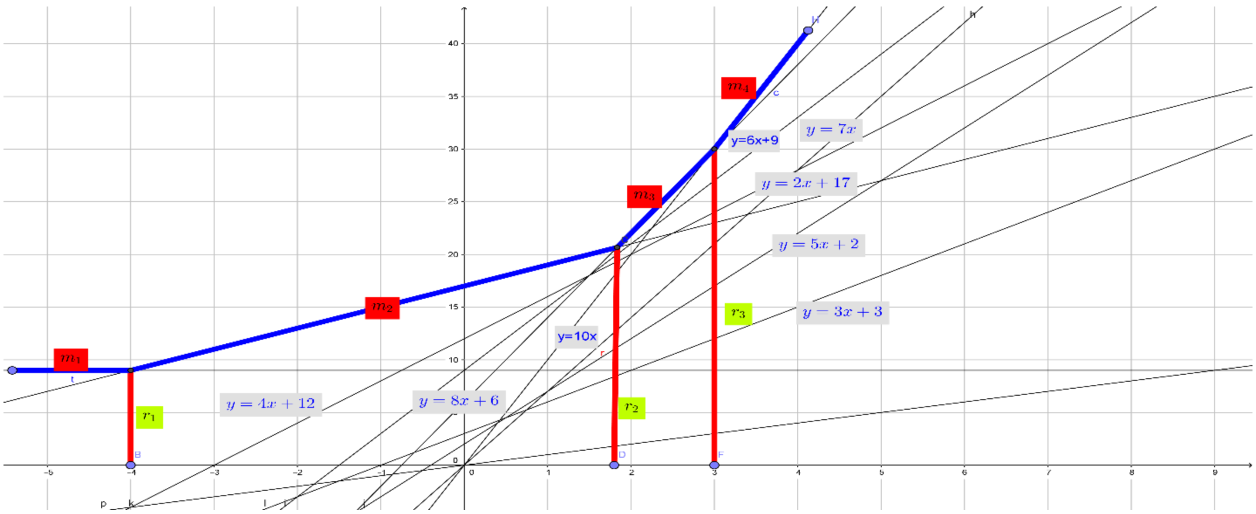

Figure I : Tropical curve of $\alpha(x)$

From Figuer I, the roots are; $r_1 = -4, r_2 = 1.7$ and $r_3 = 3$ with slopes $m_1 = 0, m_2 = 2, m_3 = 8$ and $m_4 = 10$ having multiplicities

$$M(r_1) = |m_1 - m_2| = 2$$

$$M(r_2) = |m_2 - m_3| = 6$$

$$M(r_3) = |m_3 - m_4| = 2$$

Thus, the multiplicity of $T_{\alpha(x)} = (2 \;\; 6 \;\; 2) \Rightarrow |MT_{\alpha(x)}| = 2$

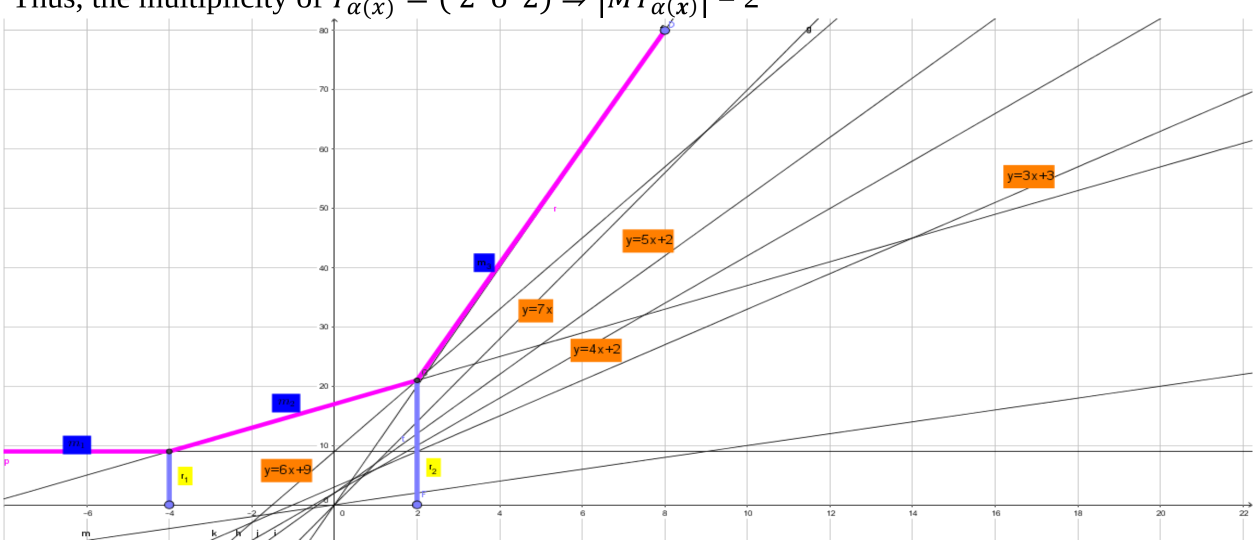

Figure II : Tropical curve of $\beta(x)$

From the tropical curve of $\beta(x)$, the roots are: $r_1 = -4,\ and\ r_2 = 2$ with slopes $m_1 = 0, m_2 = 2\ and\ m_3 = 10$ having multiplicities

$$M(r_1) = |m_1 - m_2| = 2$$

$$M(r_2) = |m_2 - m_3| = 8$$

Thus, the multiplicities of $T_{\beta(x)} = (\ 2\ \ 8\ ) \Rightarrow \left|MT_{\beta(x)}\right| = 2$.

Example 2: Consider $\gamma, \zeta \in ST_{17}$ with transformations $\gamma = (1 -1](2\ 1](3 -1](4 - 4](6\ 5\ 4]\ (7)(12)(15)(17)\ (8\ \text{-}7]\ (9\ \ 7]\ (10\ \ \text{-}8]\ (11\ \ \text{-}9]\ (\ 13\ \ 12]\ (14\ \text{-}13]\ (16\ \ 15]$ and $\zeta =$ (4)(7)(12)(15)(17) (1 -1] (2 1] (3 -1] (5 -4] (6 -5] (8 -7] (9 -7] (10 -8] (11 -9] (13 -12] (14 -13] (16 -15] having polynomial functions:

$$\gamma(x) = x^{17} + x^{16} + 4x^{15} + 3x^{14} + x^{13} + 10x^{12} + 12x^{11} + 10x^{10} + 2x^9 + 17x^8 + 3x^7 + 3x^6 + 25x^5 + 26x^4 + 2x^3 + 31x^2 + 32x + 17$$

and

$$\zeta(x) = x^{17} + x^{16} + 4x^{15} + 3x^{14} + 9x^{13} + 2x^{12} + 2x^{11} + 15x^{10} + 2x^9 + 3x^8 + 3x^7 + 3x^6 + 25x^5 + 2x^4 + 2x^3 + 31x^2 + 2x + 17$$

Then, by conditions (2) and (3) we have;

$$\mathrm{T}\gamma(x) = max\begin{Bmatrix} 17x, 16x + 3, 15x + 4, 14x + 5, 13x + 9, 12x + 10, 11x + 12, 10x + 15, 9x + 16, 8x + 17, \\ 7x + 19, 6x + 21, 5x + 25, 4x + 26, 3x + 28, 2x + 31, x + 32, 17 \end{Bmatrix}$$

$$\mathrm{T}\zeta(x) = max\begin{Bmatrix} 17x, 16x, 15x + 4, 14x + 3, 13x + 9, 12x + 2, 11x + 2, 10x + 15, 9x + 2, 8x + 3, \\ 7x + 3, 6x + 3, 5x + 25, 4x + 2, 3x + 2, 2x + 31, x + 2, 17 \end{Bmatrix}$$

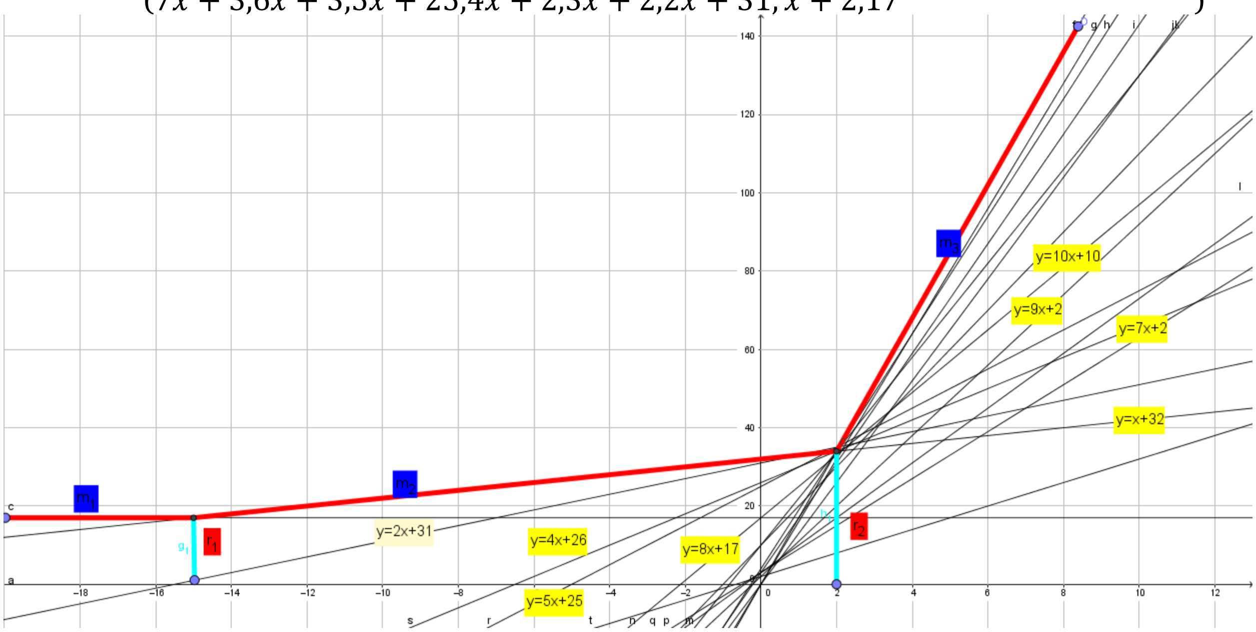

Figure III : Tropical curve of $\gamma(x)$

From Graph III, the roots are: $r_1 = -15\ and\ r_2 = 2$ with slopes $m_1 = 0, m_2 = 1\ and\ m_3 = 17$ having multiplicities

$$M(r_1) = |m_1 - m_2| = 2$$

$$M(r_2) = |m_2 - m_3| = 16$$

Thus, the multiplicities of $T_{\gamma(x)} = (\,1 \;\; 16\,) \Rightarrow |MT_{\gamma(x)}| = 2$

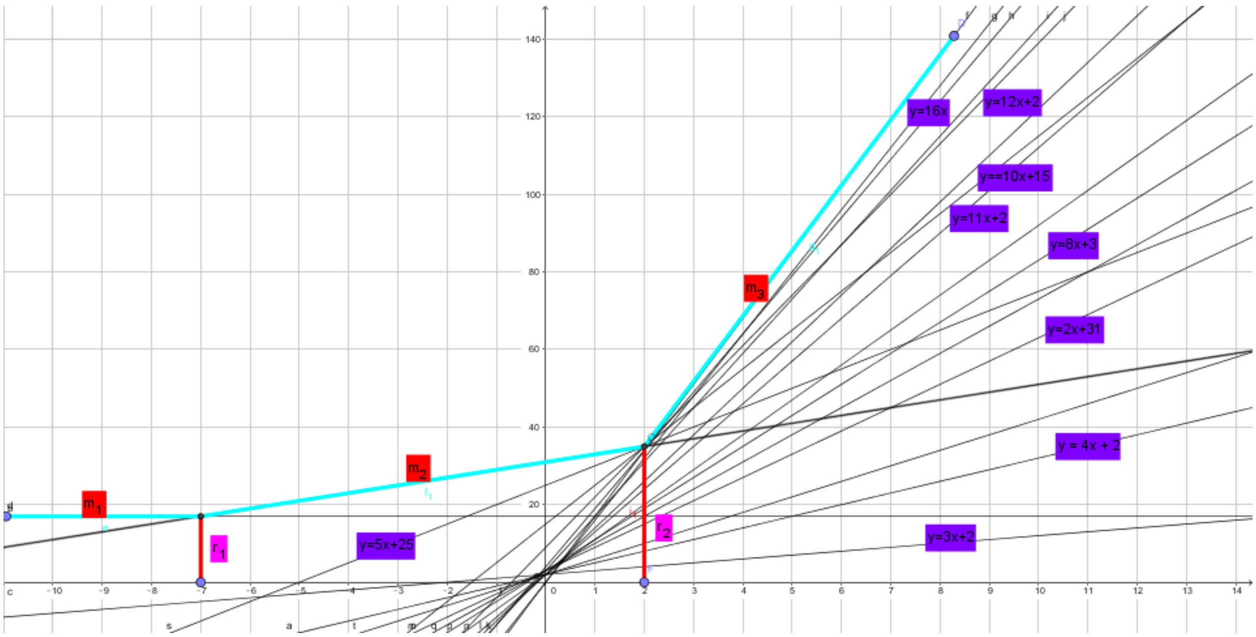

Figure IV : Tropical curve of $\zeta(x)$

From the above tropical curve of $\zeta(x)$**,** the roots are;$r_1 = -7$ *and* $r_2 = 2$ with slopes $m_1 = 0, m_2 = 2$ *and* $m_3 = 17$ having multiplicities

$$M(r_1) = |m_1 - m_2| = 2$$

$$M(r_2) = |m_2 - m_3| = 15$$

Therefore, the multiplicities of $T_{\zeta(x)} = (\,2 \;\; 15\,) \Rightarrow |MT_{\zeta(x)}| = 2.$

Hence, these results from Example 1 and Example 2 confirm the assertion in Theorem 1.

Proposition 2: Every transfomation in ST_k for $1 \leq k \leq 3$ have multiplicites of height 1.

Proof : Let the elements $\sigma, \rho, \gamma \; \epsilon \; ST_n$, where $\sigma\epsilon(ST_1)$, $\rho\epsilon(ST_2)$ *and* $\gamma\epsilon(ST_3)$ such that $\chi = \begin{pmatrix} \varphi_1 \\ \phi_1 \end{pmatrix}, \rho = \begin{pmatrix} \varphi_1 & \varphi_2 \\ -\phi_1 & \phi_2 \end{pmatrix}$ *and* $\gamma = \begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_3 \\ -\phi_1 & -\phi_2 & \phi_3 \end{pmatrix}$
with polynomial defines as follows;

$$\chi(x) = \varphi_1 x + \phi_1, \qquad \rho(x) = \varphi_1 x^2 + \varphi_2 x - \phi_1 x + \phi_2 \;\; and$$

$$\gamma(x) = \varphi_1 x^3 + \varphi_2 x^2 + \varphi_3 x - \phi_1 x^2 - \phi_2 x + \phi_3$$

Then, their tropical polynomials were given as:

$$T\chi(x) = \text{"}\varphi_1 x + \phi_1\text{"} = \max\{\varphi_1 + x, \phi_1\},$$

$$T\rho(x) = \text{"}\varphi_1 x^2 + \varphi_2 x - \phi_1 x + \phi_2\text{"} = \max\{2x + \varphi_1, x + \varphi_2, x - \phi_1, \phi_2\}$$

$$T\gamma(x) = "\varphi_1 x^3 + \varphi_2 x^2 + \varphi_3 x - \phi_1 x^2 - \phi_2 x + \phi_3"$$
$$= \max\{3x + \varphi_1,\ 2x + \varphi_2, x + \varphi_3, 2x - \phi_1, x - \phi_2,\ \phi_3\}.$$

We determined their multiplicities from the curves as:

$$|MT_{\chi(x)}| = |MT_{\rho(x)}| = |MT_{\gamma(x)}| = 1$$

■

Remark: The sum of multiplicities on every transformation in ST_k is equal to the degree of its transformaion.

Disussion of Results

Theorem 1 establishes that, every element in ST_k has the same multiplicity height for each n, which was confirmed through graphical analysis. Proposition 2 further shows that for k $\leq$ 3, all transformations in ST_k have a multiplicity of height 1, whereas for k $\geq$ 4, the multiplicity height is 2. These findings align with the algebraic closure property of the tropical semi-field, as stated by Brugalle et al. (2015). The study validates its results using examples and tropical curve representations generated through GeoGebra software.

Conclusion

In this paper, we have thoroughly analyzed the structure of the Signed Full Transformation SemigroupST_k and established that, for any given $k \geq 4$, the elements within this semigroup possess multiplicities of the same height 2 as stated in Theorem 1. Moreover, for smaller values of n where $1 \leq k \leq 3$ we have observed that, the height of the multiplicities is 1 as demonstrated in Proposition 2. This study introduces a new perspective by applying tropical techniques to the analysis of transformation semigroups, opening avenues for further research work on two other transformation semigroups specifically Partial and Partial One-to-One and their sub-semigroups.

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