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COMPARATIVE STUDY ON THE BOOTSTRAP AND JACKKNIFE METHODS FOR ESTIMATING NON-REGRESSION ESTIMATES

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ABSTRACT

This study compares the performance of Jackknife and Bootstrap resampling methods in estimating parameters of non-linear logistic growth models and the three-parameter logistic Item Response Theory (IRT) model. Both methods produced accurate estimates closely aligned with true parameter values. However, the Bootstrap method consistently demonstrated lower variance, indicating higher precision, especially under conditions of rapid or unstable growth. In the IRT model, Jackknife provided more accurate estimates for Discrimination and Difficulty parameters, while Bootstrap showed better precision overall, though with a slight tendency to underestimate Guessing and Difficulty parameters. Jackknife is preferable when unbiased estimation is critical, particularly in stable data conditions, while Bootstrap is more robust and precise in complex or volatile settings. The study recommends applying Bootstrap in high-stakes or high-variability contexts and emphasizes the importance of understanding both methods to ensure flexible and accurate data analysis in psychometric and modelling research.

1. Introduction

Regression analysis is a fundamental statistical tool used to model relationships between variables. Estimating the accuracy of regression estimates, such as coefficients and standard errors, is crucial for reliable statistical inference. Two widely used resampling methods for estimating these quantities are the Bootstrap and Jackknife methods. This study compares these methods based on their principles, efficiency, and applications in regression analysis.
The Bootstrap is a resampling method introduced by [5] in 1979, which repeatedly samples with replacement from the original dataset to create multiple resamples of the same size. For each bootstrap sample, regression estimates are recalculated, allowing the construction of confidence intervals and standard error estimates.

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The procedure involves creating a large number of bootstrap samples from a dataset of size *n*, fitting the regression model to each sample, and recording the estimated coefficients. From the distribution of these coefficients, one can derive standard errors, confidence intervals, or apply bias correction. The Bootstrap method does not rely on strong assumptions about the data distribution, making it effective for small and complex datasets. It also facilitates the estimation of bias and the construction of confidence intervals. However, it is computationally intensive due to repeated resampling and may perform poorly with dependent data or highly skewed distributions.

The Jackknife method, originally introduced by [18] in 1949 and further developed by [20] in 1958, is another resampling technique used primarily for bias reduction and variance estimation. Unlike the Bootstrap, the Jackknife systematically removes one observation at a time from the dataset and recalculates the regression estimates. For a dataset of size *n*, this yields *n* subsamples, each omitting a different observation. The regression model is then fitted to each subsample, and the mean and variance of the estimates are used to compute standard errors and bias. The Jackknife is simpler and less computationally demanding than the Bootstrap. It provides nearly unbiased estimates of variance and is particularly effective for large samples. However, it is less accurate than the Bootstrap in estimating the full sampling distribution of estimates and assumes the independence of observations. Its performance also tends to degrade with small sample sizes.

Comparison and Applications

The Bootstrap is generally preferred when dealing with small datasets, non-normal error distributions, or when precise confidence intervals for regression coefficients are needed. On the other hand, the Jackknife is better suited for large datasets or situations requiring computational efficiency. Both methods are invaluable for estimating regression coefficients and their standard errors. While Bootstrap offers greater flexibility and more accurate confidence intervals, its high computational demand can be a limitation. Conversely, the Jackknife provides faster estimates but may be less effective with small or complex data.

Several studies have contributed to understanding these methods' comparative strengths and limitations. [3] examined the effectiveness of various resampling techniques, including Bootstrap and subsampling, within high-dimensional regularized regression frameworks. Their findings indicated that while both methods are instrumental for error estimation, their performance depends on data complexity. The Bootstrap method demonstrated superior accuracy in handling complex, high-dimensional data, whereas the Jackknife was more computationally efficient in lower-dimensional contexts. [19] explored the estimation of regression parameters using Bootstrap and Jackknife resampling techniques. Their comparative analysis focused on bias, standard errors, and confidence intervals of regression coefficients, emphasizing the relative accuracy and reliability of the two methods in linear regression contexts.

[15] provided a theoretical overview of both resampling techniques, detailing their core principles, advantages, and limitations across various statistical applications. Building on this, [17] assessed the precision of Bootstrap and Jackknife methods in estimating regression parameters. Their findings suggested that while both techniques are effective, the Bootstrap approach generally yields more accurate estimates under specific conditions. [14] introduced computationally efficient methods to obtain Jackknife-based cluster-robust variance matrix estimators for linear regression models. They also proposed new variants of the wild cluster Bootstrap, demonstrating through simulations that these methods can improve inference reliability, particularly with small numbers of clusters or varying cluster sizes. On their part, [7] focused their research on determining the

most accurate method for estimating logistic regression models by comparing the Bootstrap and Jackknife techniques. Conducted with a sample of 142 individuals from Al-Hussein General Hospital, their study underscored the importance of these resampling methods in achieving precise estimates, particularly in medical statistics.

Accurate estimation of regression coefficients and their associated standard errors is central to valid statistical inference in regression analysis. Traditional methods for assessing the variability of these estimates often rely on strong parametric assumptions, such as normality and independence, which are not always met in real-world data. Resampling techniques, particularly the Bootstrap and Jackknife methods, have emerged as powerful alternatives due to their flexibility and minimal distributional requirements. However, despite their widespread use, there remains a lack of clarity regarding their relative strengths, limitations, and optimal contexts of application. Researchers and practitioners are often uncertain about which method to choose under specific data conditions, especially when dealing with small samples, nonlinearity, or high-dimensional data. This study addresses this gap by systematically comparing the Bootstrap and Jackknife methods in terms of their theoretical foundations, computational efficiency, and empirical performance in estimating regression parameters, thereby guiding more informed methodological choices in statistical analysis.

Objectives of the Study

1. To examine the principles and procedures of the Bootstrap and Jackknife resampling methods used in regression analysis.
2. To compare the Bootstrap and Jackknife methods' efficiency, accuracy, and computational demands in estimating regression coefficients and their standard errors.
3. To evaluate the suitability of the Bootstrap and Jackknife methods across different data conditions, including small and large sample sizes, and linear and nonlinear regression models.

METHODS AND MATERIALS

Jackknife in Nonlinear Regression

The Jackknife method systematically removes one observation at a time from the dataset, recalculates the regression estimate, and then aggregates the results to estimate variance and bias.

Jackknife Estimate of Nonlinear Regression Coefficients

Given a nonlinear regression model:
$Y = f(X, \theta) + \epsilon$
where:
Y is the response variable, X represents the predictor(s), θ is the parameter vector,
$\epsilon \sim N(0,\sigma^2)$ represents the error term, and $f(X, \theta)$ is a nonlinear function (e.g., exponential, logistic, or power function).
For Jackknife estimation, the procedure follows these steps:

1. Remove the ith observation and fit the nonlinear model to the remaining $n-1$ observations.
2. Obtain parameter estimates $\hat{\theta}(i)$ for each subset.
3. Compute the Jackknife estimate of θ as:

$$\hat{\theta}_{Jack} = n\frac{\hat{\theta} - (n-1)\bar{\theta}_{(i)}}{n}$$

where

$$\bar{\theta}_{(i)} = \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_i$$

The Jackknife variance is computed as:

$$Var\left(\hat{\theta}\right) = \frac{n-1}{n}\sum_{i=1}^{n}\widehat{(\theta}_{(i)} - \bar{\theta}_{(i)})^2$$

Bootstrap Estimation Procedure

1. Resample the dataset B times (with replacement) to create B bootstrap samples.
2. Fit the nonlinear model to each resampled dataset and compute the parameter estimates $\hat{\theta}$
3. Compute the Bootstrap estimate of $\hat{\theta}$ as:

$$\hat{\theta}_{BOOT} = \frac{1}{B}\sum_{b=1}^{B}\hat{\theta}^*{}_b$$

The Bootstrap variance is:

$$Var\left(\hat{\theta}\right) = \frac{1}{B-1}\sum_{b=1}^{B}(\hat{\theta}^*{}_b - \hat{\theta}_{BOOT})^2$$

Construct confidence intervals using the percentile method or the bias-corrected accelerated (BCa) method.

Nonlinear Regression Model

We shall consider the logistic growth model. A logistic growth model is a typical nonlinear regression model:

$$Y = \frac{A}{1+\exp\left(-B(X-C)\right)} + \epsilon$$

where:
A, B, and C are parameters to be estimated,
X is the independent variable,
Y is the response,
ϵ is the error term.

Applying Jackknife:

1. Remove one observation at a time and refit the logistic model.
2. Compute Jackknife estimates for A, B, and C.
3. Estimate bias and standard error using Jackknife formulas.

Applying Bootstrap:

1. Resample data B times (B = 1000).
2. Fit the logistic model for each resampled dataset.
3. Compute Bootstrap estimates for A, B, and C.

4. Construct confidence intervals using the Bootstrap distribution.

We considered two separate examples of non-linear regression using the three Logistic Growth Model and the Rash Three-parameter Model with different parameter values and performed Bootstrap and Jackknife estimations for each.

Three Logistic Growth Models

We ran three separate cases with different true parameters for the logistic growth model:

$$Y = \frac{A}{1+\exp\left(-B(X-C)\right)}$$

The Rash Three-Parameter Model

The Rasch Three-Parameter Model is a commonly used Item Response Theory (IRT) model that accounts for:

$$P(X=1|\theta) = c + (1-c)\frac{1}{1+\exp\left(-a(\theta-b)\right)}$$

where:

$P(X=1|\theta)$ is the probability of a correct response, a is the discrimination parameter, b is the difficulty parameter, c is the guessing parameter, and θ is the ability level of the examinee.

RESULTS

Logistic Growth Model Parameter Estimation

Three scenarios representing different growth dynamics, moderate growth, slower growth with a lower asymptote, and faster growth with a higher asymptote, were examined. In each case, the true parameters (A, B, C) were known and used to assess estimation performance.

Case 1: Moderate Growth Rate (A=10, B=1, C=5)

The original parameter estimates closely matched the true values. Both the Jackknife and Bootstrap methods produced estimates very similar to the original, with the Bootstrap method showing slightly lower variances. This indicates both techniques offer reliable approximations, with Bootstrap providing a marginal advantage in terms of stability.

Table 1: Parameter Estimates of Bootstrap and Jackknife for Moderate Growth Rate

Parameter	Original Estimate	Jackknife Mean	Jackknife Variance	Bootstrap Mean	Bootstrap Variance
A	9.9107	9.9109	0.0369	9.9217	0.0351
B	1.0210	1.0210	0.0060	1.0236	0.0055
C	4.9835	4.9836	0.0062	4.9870	0.0058

Case 2: Slower Growth Rate with Lower Asymptote (A=8, B=0.8, C=4)

Again, both Jackknife and Bootstrap estimates were nearly identical to the original values, and Bootstrap showed a consistent pattern of slightly lower variance. This consistency reinforces the robustness of these resampling methods, especially Bootstrap, in handling different growth profiles.

Table 2: Parameter Estimates of Bootstrap and Jackknife for Slower Growth Rate

Parameter	Original Estimate	Jackknife Mean	Jackknife Variance	Bootstrap Mean	Bootstrap Variance
A	7.8492	7.8492	0.0235	7.8510	0.0214
B	0.8289	0.8290	0.0029	0.8322	0.0029
C	3.9513	3.9513	0.0161	3.9563	0.0149

Case 3: Faster Growth Rate with Higher Asymptote (A=12, B=1.2, C=6)
The estimators continued to perform well, with minimal bias in both methods. Bootstrap estimates again had lower variances compared to Jackknife, supporting the conclusion that Bootstrap may be preferable when the sample size is small or when precision is critical.

Table 3: Parameter Estimates of Bootstrap and Jackknife for Faster Growth Rate .

Parameter	Original Estimate	Jackknife Mean	Jackknife Variance	Bootstrap Mean	Bootstrap Variance
A	12.1101	12.1102	0.0474	12.1125	0.0439
B	1.2428	1.2428	0.0039	1.2450	0.0037
C	6.0086	6.0086	0.0037	6.0093	0.0033

Rasch Three-Parameter Logistic Model

For this section, three synthetic datasets were generated under distinct parameter regimes. The original parameter estimates were obtained via Maximum Likelihood Estimation (MLE), and both Jackknife and Bootstrap methods were applied to assess estimation variability.

Case 1: Moderate Discrimination (a=1.5, b=0.5, c=0.2)
Jackknife estimates were close to the original with relatively low variances, while Bootstrap estimates, particularly for discrimination, showed a slight bias and higher variability. This highlights that discrimination estimates tend to be more volatile under Bootstrap resampling.

Table 4: Parameter Estimates of Bootstrap and Jackknife for Moderate Discrimination

Method	Discrimination (a)	Difficulty (b)	Guessing (c)
Original	3.4512	0.5153	0.2964
Jackknife	3.4317	0.5109	0.2935
Variance	2.8581	0.0608	0.0091
Bootstrap	3.2906	0.3588	0.2108
Variance	1.7259	0.0904	0.0122

Case 2: Lower Difficulty and Smaller Guessing (a=1.0, b=-0.5, c=0.15)
Jackknife again produced estimates closely aligned with the original values and had almost negligible variance for the guessing parameter. Bootstrap estimates showed more deviation, especially for the difficulty parameter, confirming that Jackknife provides more stable estimates under these conditions.

Table 5: Parameter Estimates of Bootstrap and Jackknife for Moderate Growth Rate

Method	Discrimination (a)	Difficulty (b)	Guessing (c)
Original	1.0061	-0.3858	0.3000
Jackknife	1.0072	-0.3864	0.3000

Variance	0.2239	0.1462	≈ 0
Bootstrap	1.1005	-0.6106	0.2522
Variance	0.3283	0.4301	0.0108

Case 3: High Discrimination with More Guessing (a=2.0, b=1.0, c=0.25)
This scenario revealed the highest variance in parameter estimates. Jackknife estimates were again closer to the original, while Bootstrap estimates, especially for the discrimination parameter, showed notable deviation. This suggests that as discrimination increases, estimation variance also increases, particularly for Bootstrap.

Table 6: Parameter Estimates of Bootstrap and Jackknife for High Discrimination with More Guessing

Method	Discrimination (a)	Difficulty (b)	Guessing (c)
Original	2.4808	0.9943	0.2812
Jackknife	2.4893	0.9912	0.2798
Variance	6.5765	0.1963	0.0187
Bootstrap	2.6597	0.7575	0.1855
Variance	2.4115	0.1726	0.0148

DISCUSSION OF RESULTS

The analysis for the Three Logistic Growth Models demonstrates the comparative effectiveness of the Jackknife and Bootstrap resampling methods in estimating model parameters under varying growth rate conditions. Across all three cases, the original parameter estimates, Jackknife means, and Bootstrap means were remarkably close to one another and to the true parameter values. This convergence suggests that both resampling methods yield unbiased and reliable estimates under moderate data conditions ([6]; [4]). However, the Bootstrap method exhibited a slight advantage in accuracy, as its means were marginally closer to the true values in certain instances [2].

A consistent pattern observed was the lower variance associated with Bootstrap estimates relative to those from the Jackknife. This indicates that the Bootstrap method offers enhanced precision and stability in parameter estimation. In contrast, the Jackknife method introduced slightly more variability, which may affect the consistency of estimates in applications where precision is critical. Additionally, the results reveal that estimation stability is influenced by the rate of growth. Case 2 (moderate growth) presented the lowest variance, suggesting that both methods perform optimally under these conditions. Conversely, Case 3 (faster growth) showed increased variance, indicating greater uncertainty in estimation as growth accelerates. Notwithstanding this increase in variability, the Bootstrap method maintained lower variance than the Jackknife, affirming its robustness in more volatile conditions [11].

With respect to individual parameters, both methods estimated the growth rate parameter (B) effectively, with minimal deviation from the true value. Nevertheless, the Bootstrap method consistently yielded lower variance, reinforcing its suitability for contexts requiring precise estimation [16]. Parameter C was least affected by changes in growth rate across all cases, and both methods performed equally well in estimating this parameter, with Bootstrap again producing slightly lower variance [10].

In summary, both the Jackknife and Bootstrap methods provided accurate and consistent estimates across different growth scenarios. However, the Bootstrap method was consistently superior in terms of precision, particularly in high-growth or unstable conditions. These findings support the use of Bootstrap resampling in applications where minimizing estimation variance is of primary importance [9].

Example 2: Estimation of Item Response Theory (IRT) Parameters

The analysis for the estimation of Item Response Theory (IRT) parameters, Discrimination (a), Difficulty (b), and Guessing (c), was examined using Jackknife and Bootstrap methods under three testing conditions. The results provide differentiated insights into the performance of each resampling method for the respective IRT parameters. For the Discrimination parameter (a), the Jackknife method consistently yielded means closer to the original estimates, suggesting greater accuracy and stability. However, the Bootstrap method demonstrated lower variance, indicating higher precision [13]. This trade-off is particularly relevant in Case 3, where Bootstrap slightly overestimated discrimination but retained its precision advantage. In estimating the Difficulty parameter (b), Jackknife estimates remained closer to the original values across all cases, especially in Cases 2 and 3, indicating reduced bias. In contrast, Bootstrap estimates exhibited greater deviations in these cases, suggesting susceptibility to bias under certain conditions [1]. Despite this, the Bootstrap method again outperformed in terms of variance, underscoring its precision.

For the Guessing parameter (c), both resampling methods produced plausible estimates. Nonetheless, Bootstrap tended to underestimate this parameter, particularly in Case 3. Despite this tendency, it continued to provide lower variance than the Jackknife, which supports its use when precision is prioritized [8]. Notably, the Jackknife method performed better in Case 2, where it more accurately captured low levels of guessing.

Taken together, these results suggest that while the Jackknife method generally provides more accurate estimates, particularly for Discrimination and Difficulty, the Bootstrap method consistently offers higher precision due to its lower variance. The choice of resampling method should thus align with the specific estimation objective. For applications emphasizing unbiasedness and accuracy, the Jackknife method is preferable. Conversely, in scenarios where precision and consistency are paramount, the Bootstrap method is more suitable [12].

CONCLUSION AND RECOMMENDATIONS

This study compared the performance of Jackknife and Bootstrap resampling methods in estimating parameters for non-linear regression (growth models) and the three-parameter logistic model. Both methods produced accurate estimates, but their effectiveness varied by parameter. Bootstrap consistently showed lower variance, making it more precise and preferable for applications requiring high accuracy, especially in unstable or rapidly changing growth conditions. Jackknife, while less precise, provided more unbiased estimates, particularly for difficulty and discrimination parameters in logistic models. Bootstrap tended to underestimate some parameters but remained more stable overall. Thus, Bootstrap is ideal when precision is key, while Jackknife is better for unbiased estimation in more stable settings. Understanding both methods enhances flexibility in statistical analysis.

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