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THE USE OF MULTI-ABSORBING STATES MARKOV CHAIN IN THE ANALYSIS OF NON-HOMOGENEOUS MARKOV FUZZY MANPOWER SYSTEM

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ABSTRACT

There are many studies in literature that took into consideration intra-state heterogeneity concerning individual transition behavior due to latent factors. However, none of these studies in literature has captured any specific or combination of specific latent factors responsible for differences in transition behavior within a homogeneous group in manpower model. Also, no work in literature has been able to unbundle successfully retired staff in a non-homogeneous Markov fuzzy manpower model. In this study, we considered a hierarchically graded Markov manpower system where promotion of employees is based on the innovativeness and job performance capability levels of the personnel. In this study, the model is proposed to deal with problem of vagueness involved in gradual transition of members from one grade to another in a manpower system. The model is also proposed to incorporate key personality traits that influence employees belonging to a homogeneous group to behave differently. This study also seeks to unbundle successfully retired staff in non-homogeneous Markov fuzzy manpower system using multi-absorbing states Markov chain. The mean time to absorption and long run absorption rates in each fuzzy state were obtained.

1. INTRODUCTION

Fuzzy is a word that suggests vagueness or ambiguity. According to [1] 'Fuzzy set is a set that does not have clearly defined boundaries and can contain members only at some degree'. [2] defined fuzzy set mathematically as 'A fuzzy set D on a non-empty set U is defined as a set of ordered pairs $\{(U, \mu_D(u)), u \epsilon D\}$, where $\mu_D(u)$ is the membership grade of u in D'. The set D is characterized by its membership function $\mu_D: U \to [0,1]$. A fuzzy manpower system is defined by [1] as 'a manpower system that consists of fuzzy states'. [3] defined manpower system as 'a system that consists of group of people working together for the purpose of achieving the common goal of the organization'.

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Aggregately, the workforce of any organizational manpower system comprises of a stock of heterogeneous personnel. In manpower models e.g. [4], [5], [6], manpower system is graded hierarchically into mutually exclusive and exhaustive grades where each member belongs to one and only one of the grades at any time t. The aggregated personnel system is partitioned into homogeneous groups (or grades) so that staff that belong to the same grade possess same attributes such as rank, job, age, or experience [7]. These members belonging to same grade are presumed to evolve analogously. The grades are defined based on any relevant state variable; staff flow due to recruitment (incoming flow), internal flow, as well as attrition (flow out of the system).

In organizational manpower system, it is assumed that the manpower system is partitioned into distinct classes where each member of the system clearly belongs to one and only one class at time t and transitions from one class to another at time $t+1$. Also, each member of a homogeneous group possesses same transition rate to the next higher grade. However, this assumption is not realistic in some situations concerning classification of manpower systems. Concerning some situations in manpower planning analysis, in applications of Markov theory, we are often faced with the fact that the states of the system cannot be precisely measured due to vagueness in transition of members from particular grade to another. The set of states (personnel categories) is perceived to have states with imprecise boundaries which facilitate gradual transition from membership to non-membership and vice versa.

The differences in transition behavior among members belonging to the same grade motivated this study. The differences in transition behavior are as a result of different combinations of levels of personnel traits possessed by individuals in the manpower system.

[8] worked on formulation of multilevel manpower system in discrete time homogeneous Markov model. He extended the structure of manpower system in a departmentalized framework. He further utilized the features of the extended manpower structure to create a scenario of personnel membership in three classes: the active, non-active and external classes. [9] studied the problem of ergodicity in non-homogeneous Markov system. In the study he relaxed the basic assumption present in all studies of asymptotic behavior. This assumption is that the inherent inhomogeneous Markov chain converges to a homogeneous Markov chain with regular transition probability matrix. [10] considered studying of personnel grade levels transitions in private university in Nigeria with interest in academic staff using Markovian approach, in their study, the objective is to estimate the proportion of staff recruited, promoted and withdrawn from different grades in the private university and to forecast the manpower structure in the long run. [11] devoted to partitioning personnel system based on latent factors to handle the sources of personnel differences, [12] considered that a more realistic way to describe a model is by using intervals that include the desired values of the parameters. He estimated the parameters from a data set. He considered it natural that they will be in confidence intervals and he finally studied Non-Homogeneous Markov systems process in which the desired basic parameters are in intervals. However, in real life, manpower systems possess imprecise and dynamic humanistic factors that play a significant role in their overall behaviors. Consequently, great part of the decision making takes place in a fuzzy dynamic environment. As a result, the goals, constraints and the impact of possible actions are not exactly known. [13] introduced and defined for the first time the concept of a fuzzy non-homogeneous Markov system (F-NHMS). In his study, in order to deal with vagueness associated with the estimation of transition probabilities and input probabilities in Markov systems, he combined the theory of fuzzy logic and fuzzy reasoning with theory of Markov system and then introduced the concept of a fuzzy non-homogeneous Markov system.

Moreover, [1] and [2] studied a non-homogeneous Markov fuzzy manpower system using single absorbing Markov chain for an organization whose promotion of staff is based on innovativeness and job performance capability of the individuals. In these studies, dropout, sacked staff and successfully retired staff were lumped together as wastages. However, in our study, an attempt is made to unbundle the successfully retired staff. Also, in this study, fuzzy set theory is introduced to incorporate personality traits in the analysis of manpower system in order to address the problem of differences in transition behavior among personnel belonging to the same grade which in literature is assumed equal. The dropout encapsulates those who leave the system by resignation, sack, ill health and death, while the retired staff encapsulates those who successfully retired from service either by age or years of service. It is desirable and necessary to classify the wastages for proper and effective manpower planning and prediction

Here, we present the traditional non-homogeneous Markov manpower system (NHMMS) and non-homogeneous Markov fuzzy manpower system (NHMFMS) as described by [14] and [1] respectively. For the traditional non-homogeneous Markov manpower system, according to [14], let the aggregated manpower system be classified into grades $G_i(i = 1,2,,,,K)$, where K is the highest of the hierarchical grades. Let $\{P(t)\}_{t=1}^{\infty}$ be the sequence of transition probability matrices between the grades, $\{P_0(t)\}_{t=1}^{\infty}$ the sequence of vectors of recruitment probabilities, $\{P_{K+1}(t)\}_{t=1}^{\infty}$, the sequence of probabilities of wastage from the system. $N(t) = \{N_1(t), N_2(t), \dots, N_K(t)\}$, a row vector denoting the state of the system at any time t, where $N_i(t)$ is the manpower stock in G_i at time t, $\{Q(t)\}_{t=1}^{\infty}$ is the sequence of embedded non-homogeneous Markov chain, where $Q(t) = P(t) + P_{K+1}^{'}(t)P_0(t)$ and $(.)^{'}$ denotes the transpose of the respective vector, the $ijth$ element of $Q(t)$ is given by $q_{ij}(t) = P_{ij}(t) + P_{iK+1}(t)P_{0j}(t)$ and $\{T(t)\}_{t=1}^{\infty}$, denotes the sequence of total number of personnel in the system at time t, where $\Delta T(t) = T(t+1) - T(t)$. The manpower structure at time $t+1$ is given as $N(t+1) = N(t)Q(t) + \Delta T(t)P_0(t)$,

For the fuzzy manpower system, according to [1], let $\{P_F(t)\}_{t=1}^{\infty}$ be the sequence of transition probability matrices between the fuzzy states, $\left\{P_{OF_r}(t)\right\}_{t=1}^{\infty}$ the sequence of vectors of recruitment probabilities into fuzzy states F_r, $\left\{P_{F_r0}(t)\right\}_{t=1}^{\infty}$ the sequence of vectors of wastage probabilities from fuzzy state F_r, $\{Q_F(t)\}_{t=1}^{\infty}$ is the sequence of embedded Markov chain associated with fuzzy manpower system and $N_F(t) = \left[N_{F_1}(t), N_{F_2},(t) \dots, N_{F_l}(t)\right]$ is a row vector representing the state of fuzzy manpower system at any time t, $N_{F_r}(t)$ is the expected number of members in fuzzy state $F_r(r = 1,2, \dots, l)$ at time t. The structure of fuzzy manpower s

$$N_F(t+1) = Q_f(t)N_F^{'}(t) + \Delta T(t)p_{0F}(t).$$

2. MATERIALS AND METHODS

Let $G_i(i = 1,2, \dots, K)$ be the state space for the traditional Non-Homogeneous Markov manpower system. In this study, the $ijth$ elements of $Q(t)$ is defined as $q_{ij}(t) = P_{ij}(t) + \{P_{iW_D}(t) + P_{iW_R}(t)\}P_{0j}(t)$, where $P_{iW_D}(t)$ and $P_{iW_R}(t)$ are probabilities of wastages as dropout and retirement respectively. Let $F = \{F_1, F_2, \dots, F_l\}$ be the fuzzy state space or the set of fuzzy states. The fuzzy state, F_r $(r = 1,2, \dots, l)$ is assumed to be a fuzzy set on G_i. Let $\mu_{F_r}(i): G_i \to [0,1]$ denote the membership function for the fuzzy state F_r. We also assume that F defines a fuzzy probabilistic partition on G_i so that $\sum_{r=1}^{l} \mu_{F_r}(i) = 1$

Define $\phi = \begin{bmatrix} \mu_{F_1}(1) & \cdots & \mu_{F_l}(1) \\ \vdots & \ddots & \vdots \\ \mu_{F_1}(K) & \cdots & \mu_{F_l}(K) \end{bmatrix}$ to be an $K \times l$ matrix of membership values.

We make the following definitions.

Definition 1 [15]. Let there be two fuzzy events A and B with membership functions $\mu_{F_A}(.)$ and $\mu_{F_B}(.)$ respectively. Then, the product of two fuzzy events A and B is defined as $A.B \leftrightarrow \mu_{F_{A.B}} = \mu_{F_A}.\mu_{F_B}$

Definition 2 [15]. Let there be two fuzzy events A and B with membership functions are $\mu_{F_A}(.)$ and $\mu_{F_B}(.)$ respectively. Then, the conditional probability of fuzzy event A given fuzzy event B is defined as $prob(A|B) = \frac{prob(A.B)}{prob(B)}$; $prob(B) > 0$.

2.1 Transition Probabilities Between the Fuzzy States.

Let Z_t and Z_t^f be the non-fuzzy and fuzzy states of the manpower system at time t respectively.

For the non-fuzzy states Z_t, define $n_{i.}(t) = \sum_{j=1}^{K} n_{ij}(t)$ to be the manpower stock in G_i at time t, where $n_{ij}(t)$ is the observed flow representing the number of staff in category G_i at time t that would be promoted to category G_j at time $t+1$. The transition probability is defined as, $P_{ij}(t) = prob\{Z_{t+1} = G_J | Z_t = G_i\}$. This is the probability that a member in category G_i at time t would be promoted to category G_j at time $t+1$. The maximum likelihood estimate of $P_{ij}(t)$ is defined as;

$$P_{ij}(t) = \frac{\sum_{t=1}^{T} n_{ij}(t)}{\sum_{t=1}^{T} n_{i.}(t)}. \quad (1)$$

The transition probability matrix is defined as

$$P(t) = \begin{bmatrix} P_{11}(t) & \cdots & P_{1K}(t) \\ \vdots & \ddots & \vdots \\ P_{K1}(t) & \cdots & P_{KK}(t) \end{bmatrix}, \quad (2)$$

and the embedded Markov chain is given by

$$Q(t) = \begin{bmatrix} q_{11}(t) & \cdots & q_{iK}(t) \\ \vdots & \ddots & \vdots \\ q_{K1}(t) & \cdots & q_{KK}(t) \end{bmatrix} \quad (3)$$

where the element $q_{ij}(t) = P_{ij}(t) + \{P_{iW_D}(t) + P_{iW_R}(t)\}P_{0j}(t)$. (4)

For the fuzzy states Z_t^f,

define $P_{F_rF_s}(t) = prob\left[Z_{t+1}^f = F_S | Z_t^f = F_r\right] = \frac{prob\left[Z_{t+1}^f = F_s, Z_t^f = F_r\right]}{prob\left[Z_t^f = F_r\right]}$ (5)

$$prob\left[Z_{t+1}^f = F_s, Z_t^f = F_r\right] = \sum_{i=1}^{K}\sum_{j=1}^{K} prob\left[Z_{t+1} = G_j, Z_t = G_i\right]\mu_{F_rF_s}(i,j)$$

$$= \sum_{i=1}^{K}\sum_{j=1}^{K} P_{ij}(t) prob[Z_t = G_i]\mu_{F_r}(i)\mu_{F_s}(j) \quad (6)$$

$$prob\left[Z_t^f = F_r\right] = \sum_{i=1}^{K} prob[Z_t = G_i]\mu_{F_r}(i) \quad (7)$$

$$P_{F_rF_s}(t) = \frac{\sum_{i=1}^{K}\sum_{j=1}^{K} P_{ij}(t) prob[Z_t = G_i]\mu_{F_r}(i)\mu_{F_s}(j)}{\sum_{i=1}^{K} prob[Z_t = G_i]\mu_{F_r}(i)}$$

$$P_{F_rF_s}(t) = \left(C_r(t)\right)^{-1} \sum_{i=1}^{K}\sum_{j=1}^{K} P_{ij}(t) prob[Z_t = G_i]\mu_{F_r}(i)\mu_{F_s}(j) \quad (8)$$

where $C_r(t) = \sum_{i=1}^{K} prob[Z_t = G_i]\mu_{F_r}(i)$

Then $P_F(t) = \gamma_1(t)\phi'\gamma_2(t)P(t)\phi$, by matrix notation (9)

Where $\gamma_1(t) = diag\left(\theta_{1r}(t)\right)$. is a diagonal matrix of order $l \times l$ with $\theta_{1r}(t) = \sum_{i=1}^{K} prob[z_t = G_i]\mu_{F_r}(i)$, ϕ is an $K \times l$ matrix of membership values, $\gamma_2(t) = diag\left(\theta_{2r}(t)\right)$ is a matrix of order $K \times K$ with $\theta_{2r}(t) = prob[Z_t = G_i]$, $P(t)$ is a $K \times K$ transition probability matrix between the non-fuzzy states.

Lemma 1. The sequence of the transition probability matrices $\{P_F(t)\}_{t=1}^{\infty}$ of a non-homogeneous Markov fuzzy manpower system is a sequence of sub-stochastic matrices and is given by $\{P_{F_rF_s}(t)\}_{t=1}^{\infty};\ F_r, F_s \in F$.

Proof. In order to prove this, we recall that, out of $k+1$ personnel categories, G_{K+1} is a hypothetical category representing wastage, it is only on K categories that the orthogonal partition of fuzzy sets $\{F_1, F_2, \dots, F_l\}$ is defined and using the fact that $\sum_{r=1}^{l}\mu_{F_r}(i) = 1$ and $\sum_{j=1}^{K} P_{ij}(t) \leq 1$.

$$P_{F_rF_s}(t) = \left(C_r(t)\right)^{-1}\sum_{i=1}^{K}\sum_{j=1}^{K} P_{ij}(t)prob[Z_t = G_i]\mu_{F_r}(i)\mu_{F_s}(j)$$

$$\sum_{s=1}^{l} P_{F_rF_s}(t) = \sum_{s=1}^{l}\left(C_r(t)\right)^{-1}\sum_{i=1}^{K}\sum_{j=1}^{K} P_{ij}(t)prob[Z_t = G_i]\mu_{F_r}(i)\mu_{F_s}(j)$$

$$= \left(C_r(t)\right)^{-1}\sum_{i=1}^{K}\sum_{j=1}^{K} P_{ij}(t)prob[Z_t = G_i]\mu_{F_r}(i)\sum_{s=1}^{l}\mu_{F_s}(j)$$

$= \left(C_r(t)\right)^{-1}\sum_{i=1}^{K}\sum_{j=1}^{K} P_{ij}(t)prob[Z_t = G_i]\mu_{F_r}(i)$ (since $\sum_{s=1}^{l}\mu_{F_s}(j) = 1$)

$$\sum_{r=1}^{l} p_{F_rF_s}(t) = \left(C_r(t)\right)^{-1}\sum_{i=1}^{K} prob[C_t = G_i]\mu_{F_r}(i)\sum_{j=1}^{K} P_{ij}(t)$$

$$= \left(C_r(t)\right)^{-1}C_r(t)\sum_{j=1}^{K} P_{ij}(t)$$

Then $\sum_{s=1}^{l} P_{F_rF_s}(t) = \sum_{j=1}^{K} P_{ij}(t) \leq 1$

Theorem 1 [16]. If $\{A(t)\}_{t=1}^{\infty}$ is a sequence of irreducible regular stochastic matrices and $\lim_{t\to\infty} A(t) = A$, then the product $\prod_{i=1}^{t} A^t = A^*$.

Therefore, define $P^t = prob[Z_t = G_i] = P^{(1)}Q(1,t) = P^{(1)}\prod_{i=1}^{t} Q(i)$.

Then, $\lim_{t\to\infty} P^t = P^{(1)}Q^* = P^*$, where P^* represents any row of the matrix Q^*. Note that Q^* is an irreducible regular stochastic matrix.

Therefore, $Q^* = \lim_{t\to\infty} Q^t$ (10)

$P_F = \gamma_1 \phi' \gamma_2 P \phi$ (11)

where $\gamma_1 = diag(\theta_{1r})$ is a diagonal matrix of order $l \times l$ with $\theta_{1r} = \sum_{i=1}^{K} P_i^* \mu_{F_r}(i)$, and P_i^* is the ith element of P^*, $\gamma_2 = diag(\theta_{2r})$ is a diagonal matrix of order $K \times K$, with $\theta_{2r} = P_i^*$, π and p is as defined.

2.2 Wastage Probabilities for the Fuzzy Manpower System

Let $P_{iW_D}(t) = prob[a\ member\ leaves\ as\ dropout\ at\ time\ t+1 | Z_t = G_i]$

Similarly, $P_{F_rW_D}(t) = prob\left[a\ member\ leaves\ as\ dropout\ at\ time\ t+1 | Z_t^f = F_r\right]$

$$= \frac{prob\left[a\ member\ leaves\ as\ dropout\ at\ time\ t+1, Z_t^f = F_r\right]}{prob\left[Z_t^f = F_r\right]} \quad (12)$$

$prob\left[a\ member\ leaves\ as\ dropout\ at\ time\ t+1, Z_t^f = F_r\right]$

$$= \sum_{i=1}^{K} prob[a\ member\ leaves\ as\ dropout\ at\ time\ t+1 | Z_t = G_i] prob[X_t = G_i] \mu_{F_r}(i)$$

$$= \sum_{i=1}^{K} P_{iW_D}(t) prob[Z_t = G_i] \mu_{F_r}(i) \qquad (13)$$

$$prob\left[Z_t^f = F_r\right] = \sum_{i=1}^{K} prob[Z_t = G_i] \mu_{F_r}(i) \qquad (14)$$

$$P_{F_r W_D}(t) = \left(C_r(t)\right)^{-1} \sum_{i=1}^{K} P_{iW_D}(t) prob[Z_t = G_i] \mu_{F_r}(i),$$

Then, $P_{F_r W_D}(t) = \gamma_1(t)\beta_1(t)$, where $\beta_1(t)$ is an $l \times 1$ column vector whose elements are $\theta_{3r}(t) = \sum_{i=1}^{K} P_{iW_D}(t) prob[X_t = G_i] \mu_{F_r}(i)$.

Then, $\lim_{t\to\infty} P_{F_r W_D}(t) = P_{F_r W_D} = \gamma_1 \beta_1$. (15)

where β_1 is an $l \times 1$ column vector whose element $\theta_{3r} = \sum_{i=1}^{K} P_{iW_D} p_i^* \mu_{F_r}(i)$.
Similarly, $\lim_{t\to\infty} P_{F_r W_R}(t) = P_{F_r W_R} = \gamma_1 \beta_2$ (16)

where β_2 is an $l \times 1$ column vector whose element $\theta_{4r} = \sum_{i=1}^{K} P_{iW_R} P_i^* \mu_{F_r}(i)$

2.3 Transition Probability Matrix (TPM) for the Fuzzy Manpower System with Multi-Absorbing States

This is given as

$$Q_f(t) = \begin{bmatrix} 1 & 0 & . & . & . & 0 & . & 0 & 0 & . & . & . & 0 \\ 0 & 1 & . & . & . & 0 & . & 0 & 0 & . & . & . & 0 \\ . & & & & & . & . & . & & & & & . \\ . & & & & & . & . & . & & & & & . \\ . & & & & & . & . & . & & & & & . \\ 0 & 0 & . & . & . & 0 & . & 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ \omega_{11}(t) & \omega_{12}(t) & . & . & . & \omega_{1g}(t) & . & p_{F_1F_1}(t) & p_{F_1F_2}(t) & . & . & . & p_{F_1F_l}(t) \\ \omega_{21}(t) & \omega_{22}(t) & . & . & . & \omega_{2g}(t) & . & p_{F_2F_1}(t) & p_{F_2F_2}(t) & . & . & . & p_{F_2F_L}(t) \\ . & & & & & . & . & . & & & & & . \\ . & & & & & . & . & . & & & & & . \\ . & & & & & . & . & . & & & & & . \\ \omega_{l1}(t) & \omega_{l2}(t) & . & . & . & \omega_{lg}(t) & . & p_{F_lF_1}(t) & p_{F_lF_2}(t) & . & . & . & p_{F_lF_L}(t) \end{bmatrix}$$

In canonical form, we have,

$$Q_f = \begin{bmatrix} I & 0 \\ W & P_F \end{bmatrix} \qquad (17)$$

where I is an identity matrix of transition probabilities between the absorbing states. 0 is an $g \times l$ matrix of transition probabilities from the absorbing states to the transient states. W is an $l \times g$ matrix of probabilities of absorption.

Iterative multiplication of equation (16) gives

$$Q_f^t = \begin{bmatrix} I & 0 \\ (1 + P_F + P_F^2 + \cdots + P_F^{t-1})W & P_F^t \end{bmatrix} \qquad (18)$$

$$\lim_{t\to\infty} Q_f^t = Q_f^* = \begin{bmatrix} I & 0 \\ HW & 0 \end{bmatrix} \qquad (19)$$

(since $\lim_{t\to\infty} I = 1, \lim_{t\to\infty} 0 = 0 \; and \; \lim_{\to\infty} P_F^t = 0$, ($P_F$ is a sub-stochastic matrix), where $H = 1 + P_F + P_F^2 + \cdots .. = (1 - P_F)^{-1}$ is called the fundamental matrix (FM) for the multi-absorbing states Markov chain.

2.4 Mean Time to Absorption

The average number of years before dropout or retirement from each fuzzy state is obtained using $H = (1 - P_F)^{-1}$. The total number of years a member of staff has to stay in the system before absorption (ie dropout or retirement) is $N = (1 - P_F)^{-1}e'$ (20)

where e is an $1 \times l$ row vector of ones.

2.5 Absorption Rates

Let a_r be the probability that an absorbing Markov chain will be absorbed in the absorbing fuzzy state F_s after starting in transient fuzzy state F_r. Let φ be a matrix with entries a_r. Then,

$$\varphi = HW \quad (21),$$

where $H = (1 - P_F)^{-1}$ is the fundamental matrix and $W$is as defined in the canonical form.

RESULTS

In this section, the use of multi-absorbing States Markov chain in the analysis of Non-Homogeneous Markov Fuzzy Manpower System (NMFMS) is presented. This is to illustrate the theoretical results of the previous sections. Concerning analysis of differentials in manpower systems, sources of personnel differences were classified by [17] into observable and (non observable) latent sources. It is observed that the partitioning of the manpower system into the distinct classes, G_i, is based on observable sources. [17] also classified latent sources into individual traits and environmental factors. In this study, environmental factors are restricted only to organizational culture. Also, we assume in this that the influence of organizational culture on individual career development is the same for every member of the system.

In any organizational manpower system, individual traits are very diverse. As a result, the influence of individual traits on career development is also very diverse for different members of the organization. Concerning personality study, individual traits were partitioned into five classes by ([18, [19]). They are; Openness, Conscientiousness, Extraversion, Agreeableness, and Neuroticism. In this work, we considered only Openness and Conscientiousness. [20] stated that fuzzy partitions are linguistic representations of their universe of discourse and that their elements are linguistic terms like 'low', medium', 'high'. For this study, we formulate the fuzzy partitions in terms of 'High' and 'Low' levels of the combination of the Openness and Conscientiousness. Therefore, we set $F = \{F_1, F_2, F_3, F_4\}$ to denote the fuzzy state space. F_1 is the combination of Low level of openness and Low level of conscientiousness, F_2 denotes High level of openness and Low level of conscientiousness, F_3 describes Low level of openness and High level of conscientiousness, while F_4 is High level of openness and High level of conscientiousness. We note that, the matrix ϕ of membership values is obtained based on the experts' knowledge on the system understudy [13].

Data below concerning staff promotion P_{ij}, recruitment r_{0j}, and wastage G_{K+1}) were obtained from CURTIX CABLE CONGLOMERATE, Nnewi, Anambra state, Nigeria between the year 2020 to 2023, where $G_1, G_2, G_3, G_4, and \; G_5$ represent (1) Sales Associates (2) Assistant Supervisor (3) Supervisor (4) Senior Supervisor or Assistant Managers (5) Managers respectively in hierarchical order. In the organization, promotion, as well as recruitment is done once every year. An employee is promoted if and only if he has satisfied all the promotion requirements peculiar to the initial grade set by the organization. This depends on the employee's innovative capability and job performance levels. The maximum number of years an employee can serve in the organization is 35 years and the maximum age of an employee is 65 years. Data for personnel recruitments, promotion flows, as well as wastages were collected for period of 4 years and are presented in the table below.

Table 1; Pooled staff flow based on recruitment, promotion and wastage from 2020 - 2023

1	2	3	4	5	W_D	W_R	$n_{i.}(t)$
1	262	96	0	0	0	8	0	366
2	0	300	102	0	0	12	0	414
3	0	0	158	49	0	10	5	222
4	0	0	0	122	32	7	8	109
5	0	0	0	0	68	2	12	92
r_j	15	12	16	9	2			54

From table 1, and using Eq, (1), we calculated $P(t)$, $P_{W_D}(t)$, $P_{W_R}(t)$ and $r(t)$ as

$$P(t) = \begin{bmatrix} 0.7158 & 0.2623 & 0 & 0 & 0 \\ 0 & 0.7246 & 0.2464 & 0 & 0 \\ 0 & 0 & 0.7117 & 0.2207 & 0 \\ 0 & 0 & 0 & 0.7220 & 0.1893 \\ 0 & 0 & 0 & 0 & 0.8293 \end{bmatrix}$$

$$P_{W_D}(t) = \begin{bmatrix} 0.0219 \\ 0.0290 \\ 0.0450 \\ 0.0414 \\ 0.0244 \end{bmatrix} \qquad P_{W_R}(t) = \begin{bmatrix} 0.0000 \\ 0.0000 \\ 0.0226 \\ 0.0473 \\ 0.1463 \end{bmatrix}$$

$$r(t) = \begin{bmatrix} 0.2778 & 0.2222 & 0.2963 & 0.1667 & 0.0370 \end{bmatrix}$$

Using Eq. (4), we calculated the matrix Q as

$$\lim_{t \to \cdots} Q(t) = Q = \begin{bmatrix} 0.7219 & 0.2672 & 0.0065 & 0.0036 & 0.0008 \\ 0.0081 & 0.7310 & 0.2550 & 0.0048 & 0.0011 \\ 0.0188 & 0.0150 & 0.7317 & 0.2320 & 0.0025 \\ 0.0246 & 0.0197 & 0.0263 & 0.7368 & 0.1926 \\ 0.0474 & 0.0379 & 0.0506 & 0.0285 & 0.8356 \end{bmatrix}$$

Using Theorem 1, we obtained Q^* as

$$\lim_{t\to\infty} Q^t = Q^* \begin{bmatrix} 0.0895 & 0.1591 & 0.2301 & 0.2378 & 0.2836 \\ 0.0895 & 0.1591 & 0.2301 & 0.2378 & 0.2836 \\ 0.0895 & 0.1591 & 0.2301 & 0.2378 & 0.2836 \\ 0.0895 & 0.1591 & 0.2301 & 0.2378 & 0.2835 \\ 0.0895 & 0.1591 & 0.2301 & 0.2378 & 0.2836 \end{bmatrix}$$

$$\phi = \begin{bmatrix} 0.7 & 0.1 & 0.2 & 0 \\ 0.6 & 0.1 & 0.3 & 0 \\ 0.1 & 0.4 & 0.3 & 0.2 \\ 0.1 & 0.3 & 0.1 & 0.5 \\ 0 & 0.1 & 0.1 & 0.8 \end{bmatrix}$$

To estimate the elements of γ_1, we use $\theta_{11} = \left(\sum_{i=1}^{k} p_i^* \mu_{F_1}(i)\right)^{-1} = 0.0895 * 0.7 + 0.1591 * .0.6 + 0.2301 * 0.1 + 0.2378 * 0{,}1 + 0.2836 * 0 = 4.8804.$
Others were similarly obtained, then

$$\gamma_1 = \begin{bmatrix} 4.8804 & 0 & 0 & 0 \\ 0 & 4.6168 & 0 & 0 \\ 0 & 0 & 5.3533 & 0 \\ 0 & 0 & 0 & 2.5530 \end{bmatrix}$$

$$similarly\ \gamma_2 = \begin{bmatrix} 0.0895 & 0 & 0 & 0 & 0 \\ 0 & 0.1591 & 0 & 0 & 0 \\ 0 & 0 & 0.2301 & 0 & 0 \\ 0 & 0 & 0 & 0.2378 & 0 \\ 0 & 0 & 0 & 0 & 0.2836 \end{bmatrix}$$

Therefore, using Eq.(11), we calculated the matrix P_F as

$$\lim_{t\to\infty} P_F(t) = P_F = \begin{bmatrix} 0.4341 & 0.1763 & 0.1570 & 0.1108 \\ 0.1245 & 0.2542 & 0.1589 & 0.3667 \\ 0.2240 & 0.2253 & 0.1648 & 0.2719 \\ 0.0329 & 0.1606 & 0.1033 & 0.5691 \end{bmatrix}$$

The elements of β_1, were calculated using $\theta_{31} = \sum_{i=1}^{K} P_{iW_D} P_i^* \mu_{F_1}(i) = 0.0219 * 0.0895 * 0.7 + 0.0290 * 0.1591 * 0.6 + 0.0450 * 0.2301 * 0.1 + 0.0414 * 0.2378 * 0.1 + 0.0244 * 0.2836 * 0 = 0.0061$
Other elements were similarly obtained

$$\beta_1 = \begin{bmatrix} 0.0061 \\ 0.0084 \\ 0.0065 \\ 0.0125 \end{bmatrix}$$

The elements of β_2 were similarly calculated and we have $\beta_2 = \begin{bmatrix} 0.0016 \\ 0.0095 \\ 0.0068 \\ 0.0398 \end{bmatrix}$

using Eqs. (15) and (16) respectively, we calculated the column vectors $P_{F_R W_D}$ and $P_{F_r W_R}$ as

$$P_{F_R W_D} = \begin{bmatrix} 0.0264 \\ 0..0465 \\ 0.0464 \\ 0.044 \end{bmatrix} \qquad \text{Similarly, } P_{F_r W_R} = \begin{bmatrix} 0.0116 \\ 0.0365 \\ 0.0255 \\ 0.1193 \end{bmatrix}$$

$$(1 - P_F)^{-1} = \begin{bmatrix} 3.2848 & 2.2410 & 2.0014 & 3.9888 \\ 1.5915 & 3.1102 & 1.6897 & 4.0957 \\ 1.8159 & 2.1452 & 2.8019 & 4.0343 \\ 1.2718 & 1.8839 & 1.4459 & 5.0877 \end{bmatrix}$$

Using Eqs. (20) and (21) respectively, the column vector N and matrix φ were calculated as

$$N = (1 - P_F)^{-1} e' = \begin{bmatrix} 11.5160 \\ 10.4871 \\ 10.7973 \\ 9.6393 \end{bmatrix} \qquad \varphi = \begin{bmatrix} 0.3412 & 0.6472 \\ 0.3242 & 0.6637 \\ 0.3358 & 0.6521 \\ 0.2593 & 0.7256 \end{bmatrix}$$

DISCUSSION

From the transition probability matrix (P_F) between the fuzzy states obtained, we observed that it is possible to move from one fuzzy state to another. That is, it is possible for a staff who possesses Low level of Openness and Low level of Conscientiousness to possess High level of Openness and High level of Conscientiousness at any time and so on, unlike the traditional non-homogeneous Markov manpower system where the transitions between the crisp states (grades) are only possible from the current grade to the next higher grade.

The expected length of stay in the system and the long run absorption rates in each fuzzy states were obtained. The results are somewhat unique and interesting. The expected length of stay in fuzzy state F_1 is approximately 12years and the expected length of stay in F_2 is approximately 11years years and so on. None of them is up to the maximum career length of 35years. The results show that none of the employees stays up to the maximum number of years (35years) in service. The reason could be that each fuzzy state consists of employees belonging to different grades who have different age brackets and have put up wide gap of years in service. The long run absorption rates show that employees possessing High level of openness and High level of conscientiousness have lowest rate of dropout and highest rate of retirement while employees possessing Low level of openness and Low level of conscientiousness have highest rate of dropout and lowest rate of retirement.

In literature, it is assumed that every member of manpower system belonging to the same grade has equal transition rate to the next higher grade. But in reality, this is not so. They have different transition rates. During promotion, we observed that not everybody belonging to the same grade is promoted. Most organizational manpower systems base their promotion (transition) conditions/requirements on innovative capability and job performance level. This results in employees to have different promotion behaviors although they belong to the same grade, since they have different personality traits capable of influencing their innovativeness and productivity in different ways. Also, no work in literature has considered the use of multi-absorbing states Markov chain in the analysis of fuzzy manpower systems. Previous studies [1] and [2] have used single absorbing state Markov chain in the analysis of fuzzy manpower systems where drop-out and retired staff were lumped in one absorbing state. However, to incorporate personality traits as well as dealing with the problem of ambiguity in the gradual transition of members between the crisp states of the manpower system and to unbundle the retired staff for better manpower planning, the proposed methodology in this study is highly recommended. The proposed Non-Homogeneous Markov fuzzy manpower model for modeling manpower systems will contribute functionally to the increasing literature of manpower planning based on fuzzy Markov approach. Disaggregating the population of an organization into distinct homogeneous groups is a must step for Markovian manpower planning analysis. Partitioning of aggregated manpower system into homogeneous groups (crisp states) based on the classical Markov approach still introduces vagueness concerning transition of members from one state to another. Therefore, this work will serve as a template that stimulates future readers and manpower planners to appreciate partitioning and analysis of manpower systems based on fuzzy set Markovian approach, since it has the capacity to address the vagueness introduced during gradation of the system and expresses gradual transitions from membership to non-membership and vice versa.

REFERENCES

[1] V. O. Ezugwu, A. A. Osagiede and V. A. Amenaghawon, ‘A Study on non-homogeneous Markov fuzzy manpower system’. Benin Journal of Statistics *7* (2024b) 85. http://www.bjs.uniben.org/

[2] V. O. Ezugwu, A. A. Osagiede and V. A. Amenaghawon, On the prediction of non-homogeneous Markov fuzzy manpower system. BIU Journal of Basic and Applied Sciences 9 (2024a), 53.

[3] A. U. Udom,. Optimal controllability of manpower system with linear quadratic performance index, Brazilian Journal of Probability and Statistics 28 (2014) 151. DOI:10.1214/1-BJPS195

[4] R. Belhaj and M. Tkiouat, ‘A Markov model for human resources supply forecast; dividing the HR system into subgroups’., Journal of Service Science and Management 6 (2013) 211. http://dx.doi.org/10.4236/jssm.2013.63023

[5] V. O. Ezugwu and S. Ologun, ‘Markov Chain: A predictive model for manpower Planning’, Journal of Applied Science and Environmental Management 21(2017) 557. https://dx.doi.org/10.4314/jasem.v2113.17.

[6] V. O. Ezugwu and L.I, Igbinosun, Analysis of manpower system using multi absorbing states Markov chain, International Journal of Statistics and Applied Mathematics, 5 (2020).92. http://www.mathsjournal.com

[7] T. De Feyter, ‘Modeling heterogeneity in manpower planning: dividing the personnel system into homogeneous subgroups’, Applied Stochastic models in Business and Industry 22 3 (2006) 21. DOI:10.1002/asmb.619

[8] E. O, Ossai. ‘Steady state conditions in tractable Markov manpower model for an extended manpower system’, Asian Journal of Probability and Statistics 25 (2023) 48. DOI: 10.9734/AJPAS/2023/v25i1537.

[9] P. G. Vassiliou,. ‘Strong ergodicity in non-homogeneous Markov systems with chronological order’, (2024) https://doi.org/10.3390/math12050660

[10] S. O. Agboola and H. I. Ahmad. ‘On the Markov chain prediction of academic manpower system in private university in Nigeria’, FUW Trends in Science and Technology Journal 8 (2023) 308. www.ftst.journal.com

[11] M. A. Guerry. Hidden heterogeneity in manpower systems: A Markov-switching model approach, European Journal of Operation Research 210 (2011) 106. https://doi.org/10.1016/j.ejor.2010.10.039.

[12] P. C. G. Vassiliou, ‘Non-Homogeneous Markov Set Systems’, Mathematics (2021). https//dio.org/10.3390/math9050471.

[13] M.A., Symeonaki, G.B. Stamou and S.G, Tzfestud, ‘Fuzzy non-homogeneous Markov system’, Applied Intelligence, 17 (2002) 203.

[14] P. G., Vassiliou. ‘Laws of Large Numbers for Non-Homogeneous Markov Systems’ Methodology and Computing in Applied Probabiliy (2018) https:/doi.org/10.1007/s11009-017-9612-1.

[15] M. Bhattacharrya, ‘Fuzzy Markovian decision process’, Fuzzy Sets and System 99 (1998) 273. PIIS0165-0114(96)00400-9

[16] M. A. Symeonaki, ‘Rate of convergence in fuzzy non-homogeneous Markov systems’, Communications in Statistics Theory and Methods (2017). doi:10.1080/03610926.2017.1395044.

[17] F. I. Ugwuowo, and S. I. McClean, ‘Modeling heterogeneity in manpower system: A review’. Applied Stochastic Models in Business and Industry 16, (2000).99.

[18] C. I. Thoresen, I. C. Bradley, P. D. Bliese and J. D. Thoresen, ‘The big five personality traits and individual job performance growth trajectory in maintenance and transitional job stages’, Journal of Applied Psychology 89 (2004).835. DOI:10.1037/0021-9010.89.5.835

[19] I. Ali, ‘Personality traits, individual innovativeness and satisfaction with Life’, Journal of Innovation and Knowledge, 4 (2017) 38. https://doi.org/10.1016/j.jik.2017.11.002

[20] M. A. Symeonaki and G. B. Stamou, Theory of Markov System with Fuzzy States. Fuzzy Sets and Systems 143 (2004) 427. DOI:10.1016/S0165-0114(03)00016-2.