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THE PARALOGISTIC-CHEN DISTRIBUTION: A SUBMODEL OF THE PARALOGISTIC-{- $\ln(S(x))$ } FAMILY, PROPERTIES AND APPLICATIONS

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ABSTRACT

Analysis of lifetime data is fundamental in reliability and survival studies, yet classical distributions often fail to capture complex failure-rate patterns. To address this, we introduce the Paralogistic-Chen (PCh) distribution, a new lifetime model generated via the Transformed-Transformer (T-X) method, using the Paralogistic family as a generator and the Chen distribution as a baseline. Key mathematical properties are derived and a comprehensive Monte Carlo simulation study evaluates the finite-sample and asymptotic performance of the estimators, confirming their accuracy, stability, and convergence. Parameters are estimated using maximum likelihood estimation (MLE). Applications to real datasets, along with comparisons to existing models, demonstrate that the PCh distribution provides superior goodness-of-fit and flexibility in modeling diverse lifetime behaviors. Overall, the PCh distribution offers a versatile and robust alternative for reliability and survival analysis.

1. INTRODUCTION

Probability distributions are crucial techniques in statistical modeling and inference, especially in the context of lifetime data analysis. In many real-world applications, such as engineering reliability, biomedical studies, and risk analysis, there is a growing need for flexible probability models that can accurately capture various patterns in empirical data, including increasing, decreasing, bathtub-shaped, and unimodal hazard rates.

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Most times, widely-used classical lifetime distributions often fail to adequately describe datasets with non-monotonic features. In addressing this issue, researchers have proposed generalized distributions that introduce additional parameters or utilize transformation techniques. One such powerful transformation technique is the Transformed–Transformer (*T-X*) method proposed by Alzaatreh et al. (2013), which generates family of distributions or new distributions by transforming a baseline distribution via a secondary distribution (the transformer), thereby enhancing flexibility and modeling capability.

The Transformed–Transformer (T-X) family of distributions provides a general method for constructing flexible continuous probability models. This is achieved by replacing the beta distribution in beta-generated families with the pdf, *f(t)*, of any continuous random variable and applying an additional transformation, $H[M(x)]$, to a baseline distribution, *M(x)*, where

- $H[M(x)] \in [a,b]$,
- H is differentiable and monotonically non-decreasing,
- $H[M(x)] \to a$ as $x \to 0$ and $H[M(x)] \to b$ as $x \to \infty$ x→∞.

Here, [a,b] defines the support of the transformer random variable *T*. The cumulative distribution function (cdf) of the *T–X* family is given by:

$$G(x) = \int_0^{H[M(x)]} f(t)dt = F(H[M(x)]) \tag{1}$$

where *F* and *f* are the cdf and pdf of random variable *T*. The corresponding pdf (if it exists) is

$$g(x) = \int_0^{H[M(x)]} f(t)dt = f(H[M(x)])\frac{d}{dx}H[M(x)]. \tag{2}$$

This method has since gained attraction and inspired the development of several new distributions. Alizadeh et al. (2015) introduced the Beta–Marshall–Olkin family, Bourguignon et al. (2014) proposed the Weibull–G family, and Alzaatreh et al. (2016) explored the generalized Cauchy–*Y* family. Further contributions include the Poisson–*X* family by Tahir et al. (2016) and the Quadratic–Transmuted–T–X family by Shaw and Buckley (2014). Aljarrah et al. (2014) extended the *T*–*X* approach using quantile functions, while Alzaatreh et al. (2014) introduced the Weibull–Normal{Exponential} distribution, enriching the flexibility of the *T*–Normal{*Y*} family. Nasir et al. (2019) contributed the T–Burr family, further broadening the applications of the *T–X* methodology. These studies collectively demonstrate the versatility and wide applicability of the *T–X* framework in generating new models for lifetime and reliability data. Osagie et al. (2023) developed the inverse Burr-Generalized family of distributions, derived the properties and applied a submodel of the family to illustrate the usefulness and flexibility of the new family in lifetime analysis.

Briefly, a new family of distributions shall be proposed using the paralogistic distribution as the pdf of the transformed variable *X* and $H[M(x)] = -\ln S(x)$ as the upper bound of the support of the transformer random variable *T* in (1).

The proposed generator offers distinct practical advantages over existing transformation families such as the Weibull-X and Gamma-X generators, making it particularly effective for lifetime and reliability analyses. Its single, adaptable shape parameter enables precise control over tail thickness and distributional asymmetry, allowing the model to capture both heavy-tailed and light-tailed behaviours observed in real data. Compared with other T-X transformers such as Weibull-X or Gamma-X, the paralogistic generator provides heavier and more adjustable tail behaviour through a single shape parameter. This facilitates improved modelling of datasets exhibiting extreme events or irregular hazard shapes. Furthermore, the Paralogistic-{ $-\ln S(x)$ } family produces a wide variety of hazard function shapes without requiring additional parameters. These characteristics justify the practical usefulness and modelling capability of the new generator in representing complex lifetime patterns encountered in engineering and reliability applications.

The rationale for introducing the proposed family of distributions is grounded in the need to improve the flexibility and descriptive power of statistical models used in lifetime and reliability studies. Specifically, it aims
(i) to enhance the modeling of tail behaviours in lifetime data, thereby addressing the limitations of classical distributions and offering greater adaptability in capturing tail characteristics,
(ii) to develop new distributions, such as the Paralogistic–Chen model, that can accurately to represent a wide variety of hazard patterns,
(iii) to provide improved control over tail heaviness along with an expanded range of skewness and kurtosis,
(iv) to formulate submodels with mathematically tractable forms for ease of inference and
(v) to demonstrate the applicability and robustness of the proposed distribution in modelling real-world lifetime phenomena.

Accordingly, the paper is organized into several sections. Section 2 presents the construction of the Paralogistic-$\{-\ln S(x)\}$ family as a generator and outlines key properties of the proposed family. Section 3 focuses on the Paralogistic-{Chen} distribution as a specific submodel and derives several of its statistical properties. Section 4 addresses parameter estimation and includes a simulation study for the Paralogistic-{Chen} distribution. Finally, Section 5 demonstrates the practical applicability and flexibility of the proposed model by fitting it, alongside some existing competing distributions, to two real-life datasets. Section 6 presents the conclusion to the paper.

2. METHODOLOGY

The pdf of the random variable T following a paralogistic distribution (McDonald, 1984) is defined as

$$f(t) = \vartheta^2 (1+t^{\vartheta})^{-\vartheta}, t>0, \vartheta>0\,, \qquad (3)$$

where ϑ is a shape parameter. Suppose $H[M(x)] = -\ln S(x)$ and substituting (3) in (1) defines the cdf of the Paralogistic-$\{-\ln S(x)\}$ family of distributions as

$$G_{PX}(x) = \int_0^{-\ln S(x)} \vartheta^2 (1+t^{\vartheta})^{-\vartheta} dt = 1-(1+\{-\ln S(x)\}^{\vartheta})^{-\vartheta}, x>0 \qquad (4)$$

where $S(x)$=1-$F(x)$ is the survival function of any baseline distribution.
The corresponding pdf, survival and hazard functions to (4) are given as

$$g_{PX}(x) = \vartheta^2 \frac{d\{-\ln S(x)\}}{dx} (1+\{-\ln S(x)\}^{\vartheta})^{-\vartheta-1}\,, \qquad (5)$$

$$\overline{G}_{PX}(x) = (1+\{-\ln S(x)\}^{\vartheta})^{-\vartheta}$$

and

$$h_{PX}(x) = \frac{g_{PX}(x)}{\overline{G}_{PX}(x)} = \vartheta^2 \frac{d\{-\ln S(x)\}}{dx} (1+\{-\ln S(x)\}^{\vartheta})^{-1}\,.$$

The new generator in (4) has the ability to generate new submodels from existing distributions. Some new submodels are presented in Table 1.

Table 1: Some submodels of Paralogistic-$\{-\ln S(x)\}$ family

Baseline distribution	Survival function, $S(x)$	New cdf, $G_{PX}(x)$	Submodel	Remark
Weibull	$e^{-\delta x^{\kappa}}$	$1-(1+(\delta x^{\kappa})^{\vartheta})^{-\vartheta}$	Paralogistic-Weibull	New
Lomax	$(1+\delta x)^{-\kappa}$	$1-(1+(\kappa\ln(1+\delta x))^{\vartheta})^{-\vartheta}$	Paralogistic-Lomax	New
Weibull-Exponetial	$e^{-\delta(\exp(\kappa x-1)^{\theta}}$	$1-(1+(\theta\delta(\exp(\kappa x-1))^{\vartheta})^{-\vartheta}$	Paralogistic-Weibull Exponential	New

Gompertz	$e^{-\frac{\delta}{\kappa}(\exp(\kappa x-1)}$	$1-(1+(\frac{\delta}{\kappa}(\exp(\kappa x-1))^{\vartheta})^{-\vartheta}$	Paralogistic-Gompertz	New
Burr III	$(1+x^{\delta})^{-\kappa}$	$1-(1+(\kappa\ln(1+x^{-\delta}))^{\vartheta})^{-\vartheta}$	Paralogistic-Burr III	New
Additive Weibull	$e^{-\delta x^{\kappa}-\theta x^{c}}$	$1-(1+(\delta x^{\kappa}+\theta x^{c})^{\vartheta})^{-\vartheta}$	Paralogistic-Additive Weibull	New

From Table 1, it is evident that several new distributions can be derived as submodels of Paralogistic-{-*ln S(x)*} family, provided the survival function of the baseline distribution exists. In the next section, we focus on a notable submodel of the proposed family, the Paralogistic-Chen distribution, and explore its statistical properties in detail.
The Chen distribution is chosen for its flexibility, mathematical tractability and ability to capture diverse hazard rate shapes. Its parameters have clear, interpretable effects on model behaviour, making it a suitable baseline for integration into the Paralogistic-{-*ln S(x)*} generator. Consequently, the resulting Paralogistic–Chen distribution forms a versatile and elegant submodel capable of capturing both light- and heavy-tailed behaviours within the lifetime analysis framework.

3. PARALOGISTIC-CHEN DISTRIBUTION

The Chen distribution (Chen, 2000) is a flexible lifetime distribution capable of modeling both monotonic and non-monotonic hazard functions, making it suitable for diverse applications such as modeling heart failure times, equipment lifespans, or counts of road accidents over a period (e.g., weekly, monthly, or annually). The cdf of the Chen distribution is given as

$$F(x)=1-e^{-\delta(1-e^{x^{\kappa}})^{c}},x>0,\delta,\kappa>0 \quad (6)$$

where δ and κ are the shape parameters respectively. Then, the cdf of the Paralogistic-Chen distribution is given as

$$G(x)=1-(1+(\delta(e^{x^{\kappa}}-1))^{\vartheta})^{-\vartheta},x>0,\vartheta,\delta,\kappa>0. \quad (7)$$

The corresponding pdf, survival and hazard functions of the three-parameter distribution are given as

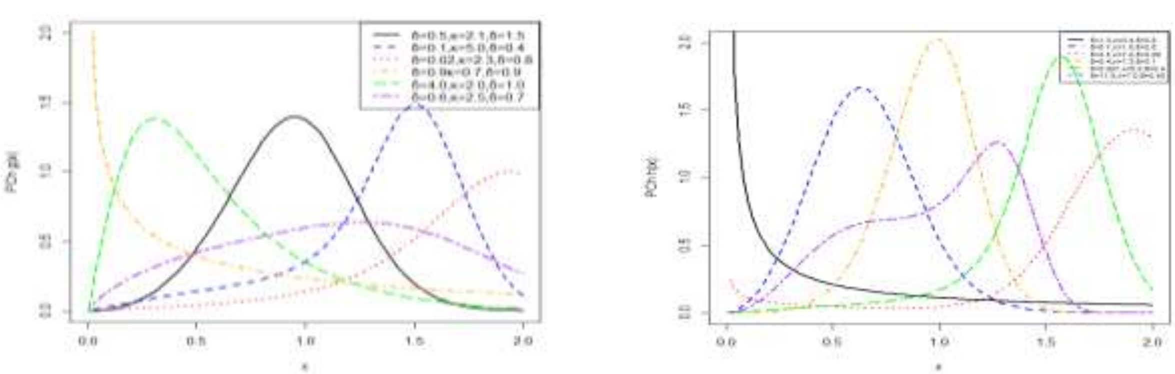
$$g(x)=\vartheta^{2}\delta\kappa x^{\kappa-1}e^{x^{\kappa}}(\delta(e^{x^{\kappa}}-1))^{\vartheta-1}(1+(\delta(e^{x^{\kappa}}-1))^{\vartheta})^{-\vartheta-1}, \quad (8)$$

$$S(x)=(1+(\delta(e^{x^{\kappa}}-1))^{\vartheta})^{-\vartheta}$$

and

$$h(x)=\vartheta^{2}\delta\kappa x^{\kappa-1}e^{x^{\kappa}}(\delta(e^{x^{\kappa}}-1))^{\vartheta-1}(1+(\delta(e^{x^{\kappa}}-1))^{\vartheta})^{-1}.$$

Plots of the pdf and hazard function for the Paralogistic-Chen distribution are shown in Figure 1.



(a) The Paralogistic-Chen pdf b) The Paralogistic-Chen hazard function

Figure 1: Plots of the probability density and hazard functions of Paralogistic-Chen distribution

Figure 1 presents selected monotonic and nonmonotonic shapes of the probability density function (pdf) and hazard function of the Paralogistic–Chen distribution. It demonstrates that the Paralogistic–Chen distribution can effectively model a wide variety of failure rate behaviors which include bimodal, increasing, decreasing, skewed, and bathtub-shaped hazard rates commonly observed in nonmonotonic lifetime data from real-world applications.

3.1 Linear expansion of the pdf of the Paralogistic-Chen distribution

The pdf of Paralogistic-Chen distribution can be expressed in series expansion. From (8), the pdf is given as

$$g(x)=\vartheta^2\delta\kappa x^{\kappa-1}e^{x^\kappa}(\delta(e^{x^\kappa}-1))^{\vartheta-1}(1+(\delta(e^{x^\kappa}-1))^{\vartheta})^{-\vartheta-1}$$

Using series expansion,

$$g(x)=\vartheta^2\delta\kappa x^{\kappa-1}e^{x^\kappa}\sum_{j=0}^{\infty}(-1)^j\binom{\vartheta+j}{j}\delta^{\vartheta(j+1)-1}(e^{x^\kappa}-1)^{\vartheta(j+1)-1}$$

$$=\vartheta^2\kappa x^{\kappa-1}\sum_{j=0}^{\infty}(-1)^j\binom{\vartheta+j}{j}\delta^{\vartheta(j+1)}e^{[\vartheta(j+1)]x^\kappa}(1-e^{-x^\kappa})^{\vartheta(j+1)-1}.$$

Substituting $e^{[\vartheta(j+1)]x^\kappa}=\sum_{l=0}^{\infty}\frac{([\vartheta(j+1)]x^\kappa)^l}{l!}$ into the above series expansion gives

$$g(x)=\vartheta^2\kappa x^{\kappa-1}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}(-1)^j\binom{\vartheta+j}{j}\frac{([\vartheta(j+1)])^l}{l!}\delta^{\vartheta(j+1)}x^{\kappa l}(1-e^{-x^\kappa})^{\vartheta(j+1)-1}.$$

Further substitution of $(1-e^{-x^\kappa})^{\vartheta(j+1)-1}=\sum_{m=0}^{\infty}/\binom{\vartheta(j+1)-1}{m}(-1)^m e^{-mx^\kappa}$ gives the linear expansion of the pdf of the Paralogistic-Chen distribution, which is given as

$$g(x)==\vartheta^2\kappa\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\binom{\vartheta+j}{j}\binom{\vartheta(j+1)-1}{m}(-1)^{j+m}e^{-mx^\kappa}\frac{([\vartheta(j+1)])^l}{l!}\delta^{\vartheta(j+1)}x^{\kappa(l+1)-1}e^{-mx^\kappa}. \quad (9)$$

3.2 Asymptotic behaviour of the Paralogistic-Chen distribution

The behavior of the proposed Paralogistic-Chen distribution is considered as $x\to0$ and $x\to\infty$. This is to determine the tail decay and type of mode the proposed distribution possesses. The pdf of the Paralogistic-Chen distribution is given in (8) as

$$g(x)=\vartheta^2\delta\kappa x^{\kappa-1}e^{x^\kappa}(\delta(e^{x^\kappa}-1))^{\vartheta-1}(1+(\delta(e^{x^\kappa}-1))^{\vartheta})^{-\vartheta-1}$$

(i) As $x\to0$, $\delta(e^{x^\kappa}-1)\approx\delta x^\kappa$ since $e^{x^\kappa}\approx x^\kappa$. It follows that $g(x)\approx\vartheta^2\delta^\vartheta\kappa x^{\kappa\vartheta-1}$.

It is seen that as $x\to0^+$,
(a) $g(x)\to0$ if $\kappa\vartheta>1$.
(b) $g(x)$ is finite, if $\kappa\vartheta=1$.
(c) $g(x)\to\infty$, if $\kappa\vartheta<1$.
From (c), the mode is near zero and $g(x)$ has a peak at the origin.

(ii) As $x\to\infty$, $(e^{x^\kappa}-1)\approx e^{x^\kappa}$ and $e^{x^\kappa}\to\infty$. It follows that $g(x)\approx\vartheta^2\delta^{-\vartheta(\vartheta+1)}\kappa x^{\kappa-1}e^{-\vartheta^2x^\kappa}\to0$.

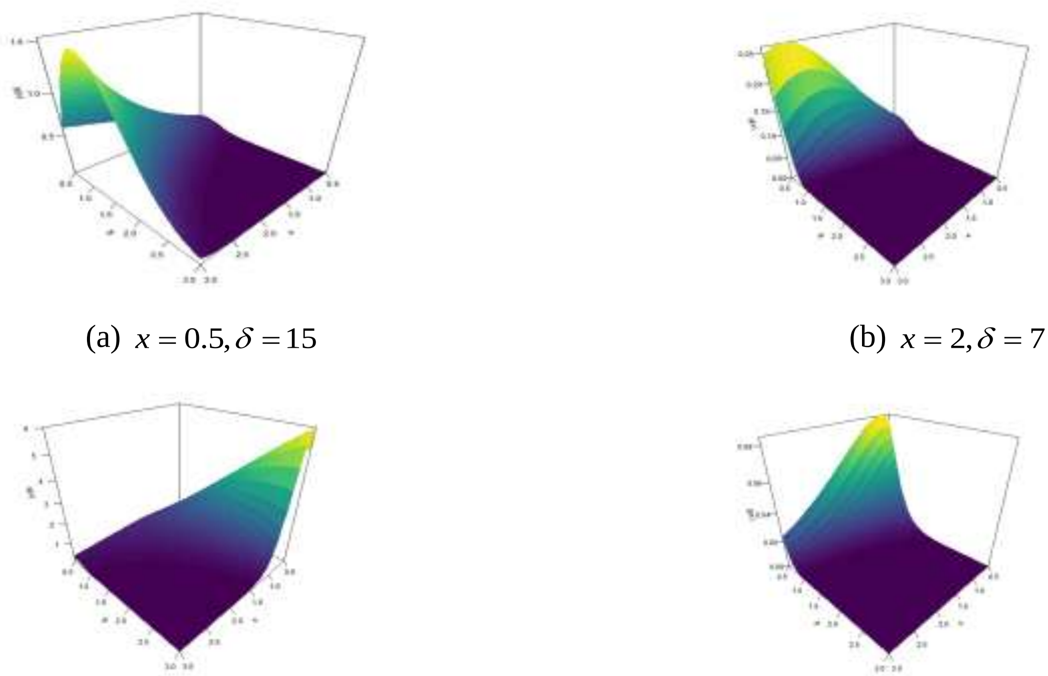
Remarks

For tail behavior of the Paralogistic-Chen distribution, the following conclusions will be made.

(a) Thinner tails are obtained if κ has larger values or ϑ has smaller values and otherwise.

(b) Thickest tails occur when κ and ϑ have large values or thinnest tails occur when κ and ϑ have smaller values.

(c) The parameters κ and ϑ are important in the modeling of extreme events, light-tailed or heavy-tailed data in lifetime analysis. This is presented in Figure 2.



(a) $x = 0.5, \delta = 15$ (b) $x = 2, \delta = 7$

(c) $x = 0.1, \delta = 2.1$ (d) $x = 0.1, \delta = 0.01$

Figure 2: Plots of the tail thickness of the pdf of the Paralogistic-Chen distribution at values of $x, \delta, \kappa, \vartheta$.

Figure 2 illustrates how the parameters of the Paralogistic–Chen (PCh) distribution influence the thickness of its tails. Tail thickness refers to how quickly the probability density declines as x increases. A thicker tail indicates a slower decay rate and therefore a higher likelihood of extreme values, while a thinner tail reflects a faster decay and fewer extremes. In plots (a) - (d), the curves show that the PCh distribution can produce both heavy and light tails depending on the parameter settings. This flexibility makes the PCh distribution suitable for modeling diverse datasets, particularly those that may exhibit significant tail behavior.

3.3 Quantile function

Theorem

If $0 < u < 1$, then the quantile function is obtained from (7) as

$$Q(u) = \left\{ \ln\left[1 + \frac{((1-u)^{-1/\vartheta} - 1)^{1/\vartheta}}{\delta} \right] \right\}^{1/\kappa} \tag{10}$$

Proof

Let $0 < u < 1$, then the quantile function of any continuous lifetime distribution is defined as $G(x_u) = u$.

Replacing $G(x)$ with the cdf of the Paralogistic-Chen distribution in (7) gives

$$1-(1+(\delta(e^{x_u^\kappa}-1))^{\vartheta})^{-\vartheta}=u.$$

After some algebraic simplification,

$$e^{x_u^\kappa}-1=\frac{((1-u)^{-1/\vartheta}-1)^{1/\vartheta}}{\delta}.$$

It follows that

$$x_u^\kappa=\left\{\ln\left[1+\frac{((1-u)^{-1/\vartheta}-1)^{1/\vartheta}}{\partial}\right]\right\}.$$

Hence, the quantile function of the Paralogistic-Chen distribution is given as

$$Q(u)=x_u=\left\{\ln\left[1+\frac{((1-u)^{-1/\vartheta}-1)^{1/\vartheta}}{\delta}\right]\right\}^{1/\kappa}.$$

The proof is complete.

Remark: It is important to note that the median of a dataset modeled by the Paralogistic-Chen distribution can be given as; $Q(0.5)=\left\{\ln\left[1+\frac{(2^{1/\vartheta}-1)^{1/\vartheta}}{\delta}\right]\right\}^{1/\kappa}$.

Table 2: Quantile values for sets of parameter values of the Paralogistic-Chen distribution

u	**$(\delta,\kappa,\vartheta)$**				
	(0.9,1.7,0.2)	**(2.3,1.8,3.5)**	**(1.7,0.2,4.0)**	**(0.06,0.3,0.9)**	**(0.6,0.5,1.0)**
0.1	0.3452908	0.3470576	0.0004574016	0.907897	0.02884598
0.2	2.1669064	0.3883210	0.0010360820	5.034477	0.12133510
0.3	3.4235976	0.4166553	0.0017577325	12.587724	0.29051763
0.4	4.4121628	0.4399700	0.0026621008	23.925058	0.55832935
0.5	5.3443804	0.4610945	0.0038001157	40.167965	0.96202275
0.6	6.3180691	0.4816921	0.0052736952	63.736385	1.56941504
0.7	7.4221378	0.5032281	0.0072763270	99.945954	2.51843801
0.8	8.8013398	0.5280902	0.0104065244	162.707691	4.14887133
0.9	10.8605792	0.5623577	0.0164632471	306.396179	7.68724822

Quantile Behaviour of the Paralogistic–Chen Distribution

Table 2 presents the quantile values of the Paralogistic–Chen distribution under various parameter settings. It is observed that the quantile function is strictly increasing and exhibits remarkable flexibility. The distribution captures a wide range of behaviours across different parameter combinations, accommodating both light and heavy tails. This versatility enables it to model skewed, concentrated, and highly dispersed data effectively, making it particularly suitable for complex lifetime and reliability analyses where classical distributions may be inadequate.

3.4 Raw moments

Theorem

If $\mu_r'=E(X^r)=\int_0^\infty x^r g(x)dx$ defines the r^{th} raw moments for any continuous lifetime distribution, then the raw moments of the Paralogistic-Chen distribution is given as

$$\mu_r' = \vartheta^2 \sum_{j=0}^{\infty}\sum_{l=0}^{\infty}(-1)^j \binom{\vartheta+j}{j} \frac{([\vartheta(j+1)])^l}{l!} \partial^{\vartheta(j+1)} \mathrm{B}\left(\frac{r}{\kappa}+l+1, \vartheta(j+1)\right). \quad (11)$$

Proof

Substituting (9) into the expression of the r^{th} raw moment be defined as $\mu_r' = \int_0^{\infty} x^r g(x)dx$,

$$\mu_r' == \vartheta^2 \kappa \sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \binom{\vartheta+j}{j}\binom{\vartheta(j+1)-1}{m}(-1)^{j+m} \frac{([\vartheta(j+1)])^l}{l!} \delta^{\vartheta(j+1)} \int_0^{\infty} x^{r+\kappa(l+1)-1} e^{-mx^{\kappa}} dx.$$

If $y = x^{\kappa}$, then

$$\mu_r' == \vartheta^2 \sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \binom{\vartheta+j}{j}\binom{\vartheta(j+1)-1}{m}(-1)^{j+m} \frac{([\vartheta(j+1)])^l}{l!} \delta^{\vartheta(j+1)} \int_0^{\infty} y^{\frac{r+\kappa(l+1)}{\kappa}-1} e^{-my} dy.$$

Hence, the raw moment for the Paralogistic-Chen distribution is given as

$$\mu_r' == \vartheta^2 \sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \binom{\vartheta+j}{j}\binom{\vartheta(j+1)-1}{m}(-1)^{j+m} \frac{([\vartheta(j+1)])^l}{l!} \delta^{\vartheta(j+1)} \Gamma\left(\frac{r}{\kappa}+(l+1)\right).$$

The proof is complete.

Figure 3: Raw moments and related quantities for Paralogistic-Chen Distribution

E(X^r)	**$(\delta,\kappa,\vartheta)$**				
	(0.5,0.8,0.6)	**(0.8,4.0,1.3)**	**(0.7,1.9,0.3)**	**(5.0,6.4,0.9)**	**(3.0,2.7,1.0)**
E(X)	3.847594	0.8957723	2.9787988	0.7914707	0.6770420
E(X²)	33.306906	0.8392883	12.6799509	0.6654619	0.5656438
E(X³)	383.642245	0.8162494	63.4545746	0.5892582	0.5536941
E(X⁴)	5154.573546	0.8197326	358.2753333	0.5457430	0.6135625
E(X⁵)	76050.697314	0.8468933	2234.1920922	0.5256287	0.7508024
SD	4.301503	0.1920426	1.9510788	0.1975755	0.3275025
CV	1.117972	0.2143877	0.6549885	0.2496308	0.4837256
CS	1.421117	-0.2307934	0.4045560	0.1002066	0.7257427
CK	4.530780	3.0676474	2.8343093	2.7351048	3.4258273

Table 3 presents the first five raw moments, along with standard deviation, coefficients of variation, skewness, and kurtosis of the Paralogistic–Chen distribution across selected parameter sets. The results illustrate the distribution’s flexibility, capturing a wide range of dispersion, skewness, and tail behavior. This adaptability makes it particularly suitable for lifetime analysis, risk assessment and reliability modeling, where control over higher-order moments is crucial.

4. PARAMETER ESTIMATION AND SIMULATION STUDY FOR PARALOGISTIC-CHEN DISTRIBUTION

4.1 Parameter Estimation

Consider a lifetime random variable, *X* modeled by the Paralogistic-Chen (PCh) distribution defined in (8). Estimation of the distribution’s parameters is based on the total log-likelihood function, which serves as the foundation for applying classical estimation methods such as Maximum Likelihood. Accurate parameter estimation is essential for capturing the underlying lifetime behavior, assessing model fit, and enabling reliable inference in applications such as reliability analysis and survival studies. Let the total log-likelihood function, $\mathrm{Z} = L(\delta,\kappa,\vartheta)$, is given by

$$\mathrm{Z} = n\ln(\vartheta^2\delta\kappa) + (\kappa-1)\sum_{i=1}^{n}\ln x_i + x_i^{\kappa} + (\vartheta-1)\sum_{i=1}^{n}\ln(\delta(e^{x_i^{\kappa}}-1)) - (\vartheta+1)\sum_{i=1}^{n}\ln(1+(\delta(e^{x_i^{\kappa}}-1))^{\vartheta}). \quad (12)$$

The partial derivatives of (12) with respect δ, κ and ϑ are used to obtain the score functions given as;

$$\frac{\partial Z}{\partial \delta} = \frac{n\vartheta}{\delta} - \vartheta(\vartheta+1)\sum_{i=1}^{n}\frac{(e^{x_i^{\kappa}}-1)(\delta(e^{x_i^{\kappa}}-1))^{\vartheta-1})}{1+(\delta(e^{x_i^{\kappa}}-1))^{\vartheta}}$$

$$\frac{\partial Z}{\partial \kappa} = \frac{n}{\kappa} + \sum_{i=1}^{n}\ln x_i + x_i^{\kappa}\ln x_i + (\vartheta-1)\sum_{i=1}^{n}\frac{x_i^{\kappa}e^{x_i^{\kappa}}\ln x_i}{(e^{x_i^{\kappa}}-1)} - \delta\vartheta(\vartheta+1)\sum_{i=1}^{n}\frac{x_i^{\kappa}e^{x_i^{\kappa}}(1+(\delta(e^{x_i^{\kappa}}-1))^{\vartheta-1}\ln x_i}{1+(\delta(e^{x_i^{\kappa}}-1))^{\vartheta}}$$

and

$$\frac{\partial Z}{\partial \vartheta} = \frac{2n}{\vartheta} + \sum_{i=1}^{n}\ln(\delta(e^{x_i^{\kappa}}-1)) - \sum_{i=1}^{n}\ln(1+(\delta(e^{x_i^{\kappa}}-1))^{\vartheta}) - (\vartheta+1)\sum_{i=1}^{n}\frac{(\delta(e^{x_i^{\kappa}}-1))^{\vartheta}\ln(\delta(e^{x_i^{\kappa}}-1))}{1+(\delta(e^{x_i^{\kappa}}-1))^{\vartheta}}.$$

The score functions, $\left(\frac{\partial Z}{\partial \delta}, \frac{\partial Z}{\partial \kappa}, \frac{\partial Z}{\partial \vartheta}\right)$, are set to zero and solved numerically to obtain the maximum likelihood estimates of the Paralogistic-Chen distribution using the *AdequacyModel* package in R software.

4.2. SIMULATION STUDY

A comprehensive Monte Carlo simulation study will be conducted to evaluate the finite-sample and asymptotic performance of three parameter estimation methods namely; the Maximum Likelihood Estimator (MLE), the Anderson-Darling Estimator (ADE), and the Minimum Product Spacing Estimator (MPSE) for the Paralogistic–Chen (PCh) distribution. Empirically, this is to validate the theoretical large-sample properties of these estimators, which are asymptotic unbiasedness and consistency. The performance of the three estimators will be assessed in terms of the bias and root mean square error (RMSE) of parameter estimates across varying sample sizes (*n* = 25, 50, 100, 250, 500, 800). Each experimental setting was replicated R = 1,000 times. Random samples were generated using the inversion method from the quantile function in (10). The essence of this study is to determine the statistical robustness and practical applicability of the PCh distribution in modeling lifetime data.

The expressions for the bias and RMSE are given as

Bias = $\frac{1}{n}\sum_{i=1}^{n}\hat{\Omega}_i - \Omega$ and RMSE = $\sqrt{\frac{1}{n}\sum_{i=1}^{n}(\hat{\Omega}_i-\Omega)^2}$,

where $\hat{\Omega}$ is the estimated parameter vector to the true parameter vector, $\Omega = (\phi, \delta, \kappa)$.

Tables 4 and 5 give the average mean estimates, bias and RMSE for the three estimators at two sets of parameter values; $\vartheta = 0.5$, $\delta = 0.1$ and $\kappa = 0.9$ and $\vartheta = 1.5$, $\delta = 3.2$ and $\kappa = 1.9$ respectively.

Table 4: Simulation study for $\vartheta = 0.5$, $\delta = 0.1$ and $\kappa = 0.9$

	n	$\hat{\vartheta}_{AV}$	$\hat{\delta}_{AV}$	$\hat{\kappa}_{AV}$	Bias(ϑ)	Bias(δ)	Bias(κ)	RMSE(ϑ)	RMSE(δ)	RMSE(κ)
MLE	25	11.6702	37887.5129	2.8433	10.1702	37884.3129	0.9433	19.2426	155366.4861	3.3979
ADE		4.4332	8583.3470	2.7384	2.9332	8580.1470	0.8384	6.6451	44179.5586	2.9370
MPSE		6.0823	14189.3037	2.3569	4.5823	14186.1037	0.4569	10.9999	89357.6820	2.6014
MLE	50	8.0464	7844.731	2.2387	6.5464	7841.531	0.3387	14.4199	67719.4860	2.2307
ADE		3.5462	2175.9673	2.2327	2.0462	2172.7673	0.3327	5.1734	19489.0734	2.0165
MPSE		4.6622	2198.3008	1.9727	3.1622	2195.1008	0.0727	8.6713	24430.8359	1.8043
MLE	100	4.0207	255.3888	2.1438	2.5207	252.1888	0.2438	8.2537	6788.7503	1.4072
ADE		2.3619	39.2761	2.1621	0.8619	36.0761	0.2621	3.0114	366.4564	1.3517
MPSE		2.7413	62.7976	1.9553	1.2413	59.5976	0.0553	4.9256	1534.1366	1.1724
MLE	250	2.1256	5.5132	1.9653	0.6256	2.3132	0.0653	3.4316	8.2344	0.7944

ADE		1.7957	6.3193	1.9624	0.2957	3.1193	0.0624	1.3667	14.7576	0.8382
MPSE		1.8179	4.5070	1.8526	0.3179	1.3070	-0.0474	1.6826	5.6806	0.7132
MLE	500	1.6023	4.0470	1.9283	0.1023	0.8470	0.0283	0.9046	3.1018	0.5363
ADE		1.5842	4.1668	1.9262	0.0842	0.9668	0.0262	0.4476	3.6332	0.5682
MPSE		1.6098	3.6221	1.8434	0.1098	0.4221	-0.0566	0.5656	2.5382	0.5090
MLE	800	1.5380	3.6477	1.9140	0.0380	0.4477	0.0140	0.2548	1.9604	0.4073
ADE		1.5456	3.6985	1.9136	0.0456	0.4985	0.0136	0.3063	2.0814	0.4330
MPSE		1.5649	3.3684	1.8480	0.0649	0.1684	-0.0520	0.2601	1.6956	0.3944

Table 5: Simulation study for $\vartheta = 1.5$, $\delta = 3.2$ and $\kappa = 1.9$

	n	$\hat{\vartheta}_{AV}$	$\hat{\delta}_{AV}$	$\hat{\kappa}_{AV}$	Bias(ϑ)	Bias(δ)	Bias(κ)	RMSE(ϑ)	RMSE(δ)	RMSE(κ)
MLE	25	5.4896	9.8869	0.8347	4.9896	9.7869	-0.0653	14.0784	140.5972	0.4595
ADE		1.8626	142.2555	0.8649	1.3626	142.1555	-0.0351	5.1136	1918.8523	0.4432
MPSE		1.9661	125.5115	0.8788	1.4661	125.4115	-0.0212	6.6348	1678.9069	0.4080
MLE	50	1.7818	0.3050	0.8615	1.2818	0.2050	-0.0385	6.5133	1.0505	0.3051
ADE		0.9361	0.4494	0.8648	0.4361	0.3494	-0.0352	2.5618	3.7466	0.2966
MPSE		0.9158	2.2252	0.8751	0.4158	2.1252	-0.0249	3.2241	55.9750	0.2780
MLE	100	0.8186	0.1487	0.8807	0.3186	0.0487	-0.0193	2.8427	0.1592	0.1994
ADE		0.5871	0.1588	0.8801	0.0871	0.0588	-0.0199	0.4812	0.2157	0.2040
MPSE		0.5823	0.1658	0.8809	0.0823	0.0658	-0.0191	1.1196	0.1916	0.1834
MLE	250	0.5184	0.1144	0.8976	0.0184	0.0144	-0.0024	0.1113	0.0549	0.1135
ADE		0.5233	0.1170	0.8939	0.0233	0.0170	-0.0061	0.1256	0.0612	0.1233
MPSE		0.5164	0.1200	0.8921	0.0164	0.0200	-0.0079	0.1084	0.0592	0.1123
MLE	500	0.5113	0.1061	0.8969	0.0113	0.0061	-0.0031	0.0770	0.0318	0.0823
ADE		0.5131	0.1076	0.8959	0.0131	0.0076	-0.0041	0.0850	0.0351	0.0896
MPSE		0.5122	0.1085	0.8917	0.0122	0.0085	-0.0083	0.0768	0.0327	0.0823
MLE	800	0.5060	0.1024	0.8995	0.0060	0.0024	-5e-04	0.0569	0.0223	0.0630
ADE		0.5072	0.1033	0.8989	0.0072	0.0033	-0.0011	0.0641	0.0237	0.0703
MPSE		0.5072	0.1038	0.8954	0.0072	0.0038	-0.0046	0.0571	0.0226	0.0631

The simulation study assessed the Maximum Likelihood (MLE), Anderson–Darling (ADE), and Minimum Product Spacing (MPSE) estimators for the Paralogistic–Chen (PCh) distribution, providing strong empirical support for their theoretical large-sample properties. Bias and RMSE decrease consistently with increasing sample size, indicating numerical stability and convergence toward true parameter values. For small samples (n = 25, 50), the estimators show substantial variability and notable bias, highlighting the challenges of reliable estimation with limited data. As sample size grows, performance stabilizes rapidly, with bias and RMSE approaching negligible levels, and all three methods effectively recover the underlying structure of the PCh distribution. Overall, the results confirm the statistical robustness and parameter uniqueness of the proposed distribution. While small-sample instability reflects its data-intensive nature, the strong large-sample performance affirms its suitability for reliability, survival analysis, and related applications requiring robust asymptotic behavior.

5. APPLICATIONS

This section presents an analysis of two lifetime datasets using the Paralogistic-Chen (PCh) distribution alongside three non-nested Chen-based models, thereby demonstrating the flexibility of the PCh as a novel parametric lifetime model. The competing distributions include the Transmuted Chen (TCh) distribution (Khan et al., 2015), the Modified Extended Chen (MECh) distribution (Anafo et al., 2022), and the original Chen (Ch) distribution (Chen, 2000). Model performance is evaluated and compared using multiple statistical criteria, including the Akaike Information Criterion (AIC), Corrected Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), Hannan–Quinn Information Criterion (HQIC), as well as the Cramér-von Mises (W*) and Anderson-Darling (A*) goodness-of-fit tests. The distribution with the best fits exhibits the lowest values of the multiple statistical criteria. The defining cdf for each competing distribution is presented as:

TCh: $G(x) = [1-\exp(\delta(1-\exp(x^{\kappa}))][(1+\vartheta)-\vartheta(1-\exp(\delta(1-\exp(x^{\kappa})))]$, $|\vartheta| \le 1, \delta > 0, \kappa > 0$.

MECh: $G(x) = [\delta(\exp(x^{\kappa})-1)+1]^{-\vartheta}$, $\vartheta > 0, \delta > 0, \kappa > 0$.

Ch: $F(x) = 1-\exp(\delta(1-\exp(x^{\kappa}))$, $\delta > 0, \kappa > 0$.

The first dataset consists of daily drought-related mortality rates in the United Kingdom, recorded from 15 April to 30 June 2020 (Mubarak & Al-Metwally, 2021). The data is given as;
0.0587, 0.0863, 0.1165, 0.1247, 0.1277, 0.1303, 0.1652, 0.2079, 0.2395, 0.2751, 0.2845, 0.2992, 0.3188, 0.3317, 0.3446, 0.3553, 0.3622, 0.3926, 0.3926, 0.4110, 0.4633, 0.4690, 0.4954, 0.5139, 0.5696, 0.5837, 0.6197, 0.6365, 0.7096, 0.7193, 0.7444, 0.8590, 1.0438, 1.0602, 1.1305, 1.1468, 1.1533, 1.2260, 1.2707, 1.3423, 1.4149, 1.5709, 1.6017, 1.6083, 1.6324, 1.6998, 1.8164, 1.8392, 1.8721, 1.9844, 2.1360, 2.3987, 2.4153, 2.5225, 2.7087, 2.7946, 3.3609, 3.3715, 3.7840, 3.9042, 4.1969, 4.3451, 4.4627, 4.6477, 5.3664, 5.4500, 5.7522, 6.4241, 7.0657, 7.4456, 8.2307, 9.6315, 10.1870, 11.1429, 11.2019, 11.4584.

Table 6 presents the parameter estimates and statistical criteria for the first lifetime dataset, enabling a comparison of the competing distributions to identify the model that provides the best fit for the dataset.

Table 6: Parameter estimates and discrepancy criteria for first lifetime dataset

Models	δ (std. error)	κ (std. error)	ϑ (std. error)	AIC	CAIC	BIC	HQIC	W*	A*
PCh	0.46307 (0.11208)	0.03770 (0.05912)	14.57134 (22.26436)	289.8485	290.1819	296.8407	292.6429	0.11221	0.75942
TCh	0.19585 (0.04975)	0.45103 (0.02951)	0.55035 (0.29730)	299.2964	299.6297	306.2886	302.0906	0.24596	1.57665
MECh	3.13427 (2.23740)	1.00772 (0.15411)	0.36982 (0.12912)	294.0187	294.3521	301.0109	296.8131	0.12201	0.87089
Ch	0.29321 (0.04453)	0.42286 (0.02814)	-	298.9822	299.1466	303.6437	300.8452	0.26644	1.70262

The second lifetime dataset comprises survival times of 121 breast cancer patients treated at a major hospital between 1929 and 1938. Muhammed et al. (2025) previously applied their proposed distribution to the dataset. The recorded survival times are as follows:
0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 1.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 4.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0.

Table 7 presents the parameter estimates and statistical criteria for the second lifetime data, enabling a comparison of the competing distributions to identify the model that provides the best fit. +

Table 7: Parameter estimates and statistical criteria for second lifetime dataset

Models	δ (std. error)	κ (std. error)	ϑ (std. error)	AIC	CAIC	BIC	HQIC	W*	A*
PCh	0.14795 (0.38999)	0.14866 (0.29609)	4.57640 (11.03925)	1165.2866	1165.4921	1173.6739	1168.6930	0.05416	0.38269
TCh	0.01285 (0.00382)	0.35070 (0.01059)	0.50235 (0.27747)	1167.8440	1168.0490	1176.2310	1171.2500	0.13412	0.91490
MECh	299.02983 (146.63222)	1.56311 (0.10303)	0.83431 (0.12623)	1195.4746	1195.6800	1203.8620	1198.8810	0.29131	1.92719
Ch	0.02145 (0.00474)	0.33870 (0.01061)	-	1167.5720	1167.6730	1173.1629	1169.8431	0.16243	1.08357

The unusual high standard errors in the analysis of the first and second datasets indicate mild parameter correlation and near-flat likelihood regions, which is a common behaviour in models with heavy right tails and sparse extreme observations.

5.1 DISCUSSION ON COMPARATIVE RESULTS

The analysis of the four competing models (PCh, TCh, MECh, and Ch) fitted to two independent lifetime datasets consistently identifies the Paralogistic-Chen (PCh) distribution as the statistically superior model. Across both datasets, PCh achieved the lowest values for all information criteria (AIC, CAIC, BIC, and HQIC), confirming its optimal balance between model complexity and goodness of fit. This superiority is strongly reinforced by the goodness-of-fit measures, where the PCh model consistently yielded the smallest Anderson-Darling (A*) and Cramér–von Mises (W*) statistics, indicating the closest agreement with the empirical data distribution. While the Ch and TCh models showed moderate performance, and the MECh model proved competitive in one dataset, their overall fit remained inferior to that of the PCh model. The unified results from both the information criteria and discrepancy tests underscore the PCh distribution's exceptional accuracy and flexibility as a robust alternative for lifetime data analysis.

CONCLUSION

This study introduced and comprehensively analyzed the Paralogistic–Chen (PCh) distribution as a flexible model for lifetime data. Through rigorous derivations, its mathematical properties were established, and maximum likelihood estimation was applied for parameter inference. The simulation study confirmed the numerical stability of the MLE, ADE, and MPSE estimators, with decreasing bias and RMSE across sample sizes. Real-data applications further demonstrated the

PCh model's superiority over existing Chen-based competitors. These findings highlight the value of the paralogistic generator as a flexible and practically useful tool in lifetime modelling
The comparative analysis revealed that the PCh distribution consistently outperformed some well-known competing models, achieving the lowest information criteria values and superior goodness-of-fit measures. These results demonstrate that the PCh distribution is a flexible and adaptable modeling choice, effectively capturing diverse data behaviors with minimal parameter complexity. The ability of the distribution to model lifetime datasets with monotonic and nonmonotonic characteristics underscores its potential for broad applications in reliability analysis, biomedical studies, and other applied statistical domains. Future work may explore Bayesian estimation techniques, regression extensions, and multivariate generalizations to further enhance its utility.

AUTHORS CONTRIBUTIONS

Musa conceived and developed the introduction, while Osagie performed the dataset analyses. Both authors jointly contributed to the methodology and parameter estimation, and collaboratively reviewed and refined the manuscript in its entirety.

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DECLARATION OF INTEREST

The authors declare no conflict of interest.

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