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ON INVERSES OF THIRD HANKEL DETERMINANT INVOLVING SYMMETRIC AND CONJUGATE POINTS

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ABSTRACT

After thorough investigations, the authors observed that the third-order Hankel determinant for inverses associated with the symmetric and conjugate points have not appeared in print. This prompted the authors to investigate the third-order Hankel determinant for inverses for the classes S_s^* and S_c^* using Gelova and Tuneski's approach. This work establishes new upper bounds for the third-order Hankel determinant for these classes of inverse functions.

1. INTRODUCTION

Let A denote the family of the functions

$$k(z) = z + \sum_{t=2}^{\infty} a_t z^t$$

which are analytic in the open unit disk $\psi = \{z: |z| < 1\}$ with normalization conditions $K(0) = 0$ and $K'(0) - 1 = 0$. Also note, that $S \subset A$ is a class of Univalent function with the same normalization conditions mentioned above. Several researchers like Fadipe et al. [[5]], Balabola [[2]], Oladipo et al. [[12]], Opoola et al. [[14]], among others, have used the series (1.1) to define many subclasses of analytic functions like starlike function, convex function, bounded turning point, with geometrical representation Re $\left(\frac{zk'(z)}{k(z)}\right) > 0$, Re $\left(1 + \frac{zk''(z)}{k'(z)}\right) > 0$, Re $k'(z) > 0$ just to mention but few and their results are widely available in the literature. Sakaguchi [[18]] used starlike function to define a starlike function with respect to symmetric points denoted by S_s^* which satisfies

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$$\operatorname{Re}\left\{\frac{zk'(z)}{k(z)-k(-z)}\right\} > 0 \quad z \in \psi \tag{1.2}$$

while Ashwah and Thomas [[4]] used the same approach to introduce another class of analytic functions denoted by S_c^* named starlike with respect to conjugate points

$$\operatorname{Re}\left\{\frac{zk'(z)}{k(z)+\overline{k(\bar{z})}}\right\} > 0, \quad z \in \psi \tag{1.3}$$

Equations (1.2) and (1.3) have been used to define several subclasses of analytic functions in different ways with different perspectives and their results have been published extensively. For details see Selvaranj and Vasanthi [[19]], Olatunji and Oladipo [[13]] and so on.

The Hankel determinant $H_q(t)(k)$ for a given function k with $q \geq 1$ and $n \geq 1$ of the form

$$H_q(t)(k) = \begin{vmatrix} a_t & a_{t+1} & \cdots & a_{t+q-1} \\ a_{t+1} & a_{t+2} & \cdots & a_{t+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{t+q-1} & a_{t+q} & \cdots & a_{t+2q-2} \end{vmatrix} \tag{1.4}$$

is the so-called Hankel determinant Setting $q = 3$ and $t = 1$ in (1.4), we obtain

$$H_3(1)(k) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2) \tag{1.5}$$

which is referred to as the third-order Hankel determinant.

Khatter et al. [[8]] used (1.5) to study the third-order Hankel determinant of starlike and convex functions while Lecko et al. [[9]] obtained the sharp bound of the Hankel determinant of the third kind for starlike functions of order $1/2$. Rath et al. [[16]] investigated the sharp bound of the third Hankel determinant for starlike functions of order $1/2$. The upper bounds of the third Hankel determinant for classes of univalent functions with bounded turning was considered by Obradovic and Tuneski [[10]].

For every univalent function in ψ, its inverse exists at least on a disk of radius $1/4$. If the inverse has an expansion

$$k^{-1}(\omega) = \omega + A_2\omega^2 + A_3\omega^3 + \cdots \tag{1.6}$$

then, by using the identity $k(k^{-1}(\omega)) = \omega$, from (1.1) and (1.6), we have

$$\begin{aligned} A_2 &= -a_2, \\ A_3 &= -a_3 + 2a_2^2, \\ A_4 &= -a_4 + 5a_2a_3 - 5a_2^3, \\ A_5 &= -a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_2^4. \end{aligned} \tag{1.7}$$

Using (1.5) and the relations in (1.7), after simplification, we obtain

$$\begin{aligned} H_3(1)(k^{-1}) and &= A_3(A_2A_4 - A_3^2) - A_4(A_4 - A_2A_3) + A_5(A_3 - A_2^2) \\ and &= a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2) - (a_3 - a_2^2)^3 \\ and &= H_3(1)(k) - (a_3 - a_2^2)^3 \end{aligned} \tag{1.8}$$

which represents the third-order Hankel determinant for the inverse functions.

Ahmed and Roy [[1]] checked the third-order Hankel determinant for inverse coefficients of starlike functions of order $\frac{1}{2}$. In 2022 precisely, Obradovic and Tuneski [[11]] looked at third-order Hankel determinant for inverse functions of certain classes of univalent functions. Also, Raza et al. [[17]] investigated the third-order Hankel determinant for inverse coefficients of convex functions and while Gelova and Tuneski [[6]] considered the third-order Hankel determinant for inverse functions of a class of starlike functions of order α.

While there are many results on the third-order Hankel determinant for inverse functions, beyond the references cited here, no researcher has considered the third-order Hankel determinant for inverse functions in terms of symmetric and conjugate points. In this work, the authors are motivated by earlier work and use the approach of Gelova and Tuneski [[7]] to establish the third-order Hankel determinant for inverse functions associated with starlike symmetric and starlike conjugate points respectively.

In the light of this, the equations (1.2) and (1.3) will be used to obtain our results with the following lemmas.

Lemma 1 (Prokhorov and Szynal [[15]]): Let $w(z) = d_1 z + d_2 z^2 + d_3 z^3 + \cdots$ be a Schwarz function. Then, for any real number a and b such that $(a, b) \in D_1 \cup D_2$, where

$$D_1 = \{(a, b): |a| \geq 1, -1 \leq b \leq 1\}$$

and

$$D_2 = \{(a, b): \frac{1}{2} \leq |a| \leq 2, \frac{4}{27}(|a| + 1)^3 - (|a| + 1) \leq b \leq 1\},$$

the following sharp estimate holds: $|d_3 + a d_1 d_2 + b d_1^3| \leq 1$.

Lemma 2 [[3]]: Let $w(z) = d_1 z + d_2 z^2 + \cdots$ be a Schwarz function. Then $|d_2| \leq 1 - |d_1|^2$, and $|d_4| \leq 1 - |d_1|^2 - |d_2|^2 - |d_3|^2$.

MAIN RESULTS

Theorem 2.1. If $k(z) \in \mathbf{S}_s^*$ is of the form $k(z) = z + a_2 z^2 + a_3 z^3 + \cdots$, then

$$|H_3(1)(k^{-1})| \leq \frac{17}{32} = 0.53125 \ldots$$

Proof. Observe that (1.2) is equivalent with

$$k'(z)\left(\frac{2z}{k(z) - k(-z)}\right) = \frac{1 + w(z)}{1 - w(z)},$$

or

$$k'(z)[1 - w(z)] = \frac{(k(z) - k(-z))}{2z}[1 + w(z)] \qquad (2.1)$$

where w is the Schwarz function with the properties $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{D}$. If $w(z) = d_1 z + d_2 z^2 + \cdots$, then equating the coefficients in (2.1) yields

$$\begin{aligned} a_2 &= d_1, \\ a_3 &= d_1^2 + d_2, \\ a_4 &= \tfrac{1}{2}(d_3 + 3d_1 d_2 + 2d_1^3), \\ a_5 &= \tfrac{1}{2}(d_4 + 2d_1 d_3 + 5d_1^2 d_2 + 2d_1^4 + 2d_2^2). \end{aligned} \qquad (2.2)$$

Using (1.8) and (2.2), after simplification we obtain

$$\begin{aligned} H_3(1)(k^{-1}) = &\left(-\frac{1}{2}d_1^4 d_2 + \frac{1}{2}d_1^3 d_3 - \frac{3}{4}d_1^2 d_2^2 + \frac{1}{2}d_1 d_2 d_3 - d_2^3\right) \\ &-\left(\frac{1}{2}d_1^4 d_2 + \frac{1}{2}d_1^3 d_3 + \frac{3}{4}d_1^2 d_2^2 + d_1 d_2 d_3 + \frac{1}{4}d_3^2\right) \\ &+\left(d_1^4 d_2 + \frac{1}{2}d_4 d_2 + d_1 d_2 d_3 + \frac{5}{2}d_1^2 d_2 + d_2^3\right) - d_2^3. \end{aligned}$$

This simplifies to

$$H_3(1)(k^{-1}) = \frac{1}{4}(d_3(d_3 + 2d_1d_2) - 4d_2^3 + 4d_1^2d_2^2 + 2d_2d_4)$$

$$|H_3(1)(k^{-1})| \leq \frac{1}{4}(|d_3||d_3 + 2d_1d_2| + 4|d_2|^3 + 4|d_1|^2|d_2|^2 + 2|d_2||d_4|). \quad (2.3)$$

Applying lemma 1 with $a = 2, b = 0$ where $(2,0) \in D_2$, we have

$$|H_3(1)(k^{-1})| \leq \frac{1}{4}(|d_3| + 4|d_2|^3 + 4|d_1|^2|d_2|^2 + 2|d_2||d_4|).$$

Applying lemma 2, we get

$$|H_3(1)(k^{-1})| \leq \frac{1}{4}[|d_3| + 4|d_2|^2(1 - |d_1|^2) + 4|d_1|^2|d_2|^2 + 2(1 - |d_1|^2)(1 - |d_1|^2 - |d_2|^2 - |d_3|^3)]$$

This further simplifies to

$$|H_3(1)(k^{-1})| \leq \frac{1}{4}(2 + |d_3| - 2|d_3|^2 + 2|d_2|^2(1 - |d_1|^2) - 2|d_1|^2(1 - |d_3|^2) - |d_1|^2(2 - 2|d_1|^2)).$$

Furthermore, applying lemma 2 in the last equation above, it follows that the terms involving d_1 *and* d_2 are non-positive. Consequently, we obtain

$$|H_3(1)(k^{-1})| \leq \frac{1}{4}(2 + |d_3| - 2|d_3|^2).$$

For $0 \leq |d_3| \leq 1$, maximum is attained at $|d_3| = \frac{1}{4}$, giving

$$|H_3(1)(k^{-1})| \leq \frac{17}{32} = 0.53125. \quad ■$$

We conjecture that the bound in Theorem 2.1 is not sharp.

Theorem 2.2. If $k(z) \in \mathbf{S}_\mathbf{c}^*$ is of the form $k(z) = z + a_2z^2 + a_3z^3 + \cdots$, then

$$|H_3(1)(k^{-1})| \leq \frac{97}{162} = 0.598765 \ldots$$

Proof. For each $k(z) \in \mathbf{S}_\mathbf{c}^*$, there exists a Schwarz function $w(z) = d_1z + d_2z^2 + \cdots$, analytic in $\mathbb{D}$, $w(0) = 0$, $|k(z)| < 1$, such that

$$k'(z)(\frac{2z}{k(z) + \overline{k(\bar{z})}}) = \frac{1 + w(z)}{1 - w(z)},$$

or

$$k'(z)[1 - w(z)] = \frac{(k(z) + \overline{k(\bar{z})})}{2z}[1 + w(z)] . \quad (2.4)$$

From this, equating coefficients gives

$$\begin{aligned} a_2 &= 2d_1, \\ a_3 &= 3d_1^2 + d_2, \\ a_4 &= \frac{2}{3}(d_3 + 5d_1d_2 + 6d_1^3), \\ a_5 &= \frac{1}{6}(3d_4 + 14d_1d_3 + 43d_1^2d_2 + 30d_1^4 + 6d_2^2). \end{aligned} \quad (2.5)$$

Using (1.8) and (2.5), after simplification we get

$$H_3(1)(k^{-1}) = (\tfrac{1}{6}d_1^4d_2 + \tfrac{1}{3}d_1^3d_3 - \tfrac{11}{18}d_1^2d_2^2 + \tfrac{5}{9}d_1d_2d_3 - \tfrac{4}{9}d_3^2 - \tfrac{1}{2}d_1^4d_4 + \tfrac{1}{2}d_2d_4 - (-d_1^2 + d_2)^3)$$
$$= \tfrac{1}{18}[18d_1^6 - 51d_1^4d_2 + 43d_1^2d_2^2 + 6d_1^3d_3 + 10d_1d_2d_3 - 18d_2^3 - 8d_3^2 - 9d_1^2d_4 + 9d_2d_4]$$
$$= \tfrac{1}{18}\left[18d_1^6 - 8d_3\left(d_3 - \tfrac{5}{4}d_1d_2 - \tfrac{3}{4}d_1^3\right) - 51d_1^4d_2 - 18d_2^3 + 43d_1^2d_2^2 + 9(d_2 - d_1^2)d_4\right].$$

Now,

$$|H_3(1)(k^{-1})| \le \tfrac{1}{18}[18|d_1|^6 + 8|d_3||d_3 - \tfrac{5}{4}d_1d_2 - \tfrac{3}{4}d_1^3| + 51|d_1|^4|d_2| + 18|d_2|^3 + 43|d_1|^2|d_2|^2 + 9(|d_2| - |d_1|^2)|d_4|]. \quad (2.6)$$

Again, applying lemma 1 with $a = \frac{5}{4}$ and $b = \frac{3}{4}$, where $(\frac{5}{4}, \frac{3}{4}) \in \mathbb{D}_2$ and also the inequality for the function ω given in lemma 2 to (2.6) we have

$$|H_3(1)(k^{-1})| \le \frac{1}{18}[18|d_1|^6 + 8|d_3| + 51|d_1|^4|d_2| + 18|d_2|^2(1 - |d_1|^2) + 43|d_1|^2|d_2|^2 + 9(1 - 2|d_1|^2)(1 - |d_1|^2 - |d_2|^2 - |d_3|^2)].$$

After simplification, we obtain

$$|H_3(1)(k^{-1})| \le \frac{1}{18}[9 + 8|d_3| - 9|d_3|^2 - 9|d_1|^2(1 - 2|d_3|^2) + 43|d_2|^2(\frac{18}{43} - |d_1|^2) - 9|d_2|^2(1 - 2|d_1|^2) - |d_1|^2(18 - 51|d_1|^2|d_2| - 18|d_1|^2 - 18|d_1|^4)].$$

Furthermore, by applying lemma 2, it follows that the terms involving d_1 *and* d_2 are non-positive. Consequently, we obtain

$$|H_3(1)(k^{-1})| \le \frac{1}{18}[9 + 8|d_3| - 9|d_3|^2 - 9|d_1|^2(1 - 2|d_3|^2)]$$

Since $0 \le |d_3| \le 1$, we have

$$|H_3(1)(k^{-1})| \le \frac{97}{162} = 0.598765 \ldots \quad \blacksquare$$

where maximum is attained for $|d_3| = \frac{4}{9}$.

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