Transactions of
The Nigerian Association of
Mathematical Physics
Journal homepage: https://nampjournals.org.ng

OPTIMAL INVESTMENT AND CONSUMPTION STRATEGIES WITH DEBT RATIO IN A STOCHASTIC VOLATILITY MARKET

OGBUAGU, C. E.[1*], EKANG, I. F.[2], SENGE, I. O.[1], AND NKEKI, C. I.[1]
[1]*Department of Mathematics, University of Benin, Benin City, Nigeria*
[2]*Department of Mathematics, University of Uyo, Uyo, Nigeria*

ARTICLE INFO

Article history:
Received xxxxx
Revised xxxxx
Accepted xxxxx
Available online xxxxx

Keywords:
Optimal investment,
Optical consumption,
Debt ratio,
Stochastic volatility,
Housing asset,
Hamilton-Jacobi-Bellman equation.

ABSTRACT

This paper investigates an optimal investment-consumption problem with endogenous leverage in a continuous-time market characterized by stochastic volatility and housing risk. We develop a unified framework in which an investor allocates wealth among a risk-free asset, a risky financial asset with Heston-type stochastic volatility, and a non-financial housing asset financed partly through debt. The investor simultaneously chooses the portfolio allocation, consumption rate, and debt ratio to maximize expected discounted utility under constant relative risk aversion preferences. Using stochastic dynamic programming, we derive the associated Hamilton-Jacobi-Bellman equation and obtain explicit closed-form solutions for the optimal portfolio policy, optimal debt ratio, and optimal consumption rule. The analytical results reveal that stochastic volatility affects investment decisions not only directly through the financial asset but also indirectly through leverage and housing exposure. Numerical illustrations demonstrate the sensitivity of the optimal policies to changes in volatility persistence, correlation structures, borrowing costs, and risk preferences.

1 INTRODUCTION

Optimal investment and consumption decisions lie at the core of continuous–time financial economics. Since the works of Merton [1, 2], stochastic control techniques have been widely employed to characterize optimal portfolio allocation and consumption strategies under uncertainty. In these classical formulations, investors dynamically allocate wealth between risky and risk-free assets so as to maximize expected lifetime utility, typically under simplifying assumptions such as constant volatility, frictionless markets, and absence of leverage considerations.

[*]Corresponding author: OGBUAGU, C.E.
E-mail address: chukwuka.ogbuagu05@gmail.com
https://doi.org/10.60787/tnamp.v23.628

Subsequent research has substantially extended the Merton framework. Several studies incorporated borrowing and consumption constraints (Fleming and Zariphopoulou [3]; Vila and Zariphopoulou [4]), transaction costs (Dumas and Luciano [5]; Shreve and Soner [6]), and finite investment horizons (Liu and Loewenstein [7]; Dai et al. [8]). Despite their analytical richness, these models largely maintained the assumption of constant asset price volatility and often abstracted from household balance-sheet structure.

However, empirical evidence strongly suggests that financial market volatility is neither constant nor deterministic. Asset returns exhibit volatility clustering, mean reversion, and leverage effects that are inconsistent with constant-volatility models. To address these empirical shortcomings, stochastic volatility models – most notably the Heston [9] framework – have been introduced, allowing volatility itself to evolve as a stochastic process. Building on this development, several authors have examined optimal portfolio and consumption problems under stochastic volatility (Fleming and Hernandez-Hernandez [10]; Kraft [11]; Chacko and Viceira [12]; Liu [13]). These studies demonstrate that time-varying volatility materially alters optimal portfolio weights and hedging demands.

Parallel to developments in stochastic volatility, another strand of the literature has emphasized the role of leverage, borrowing, and debt dynamics in investment decisions. Debt financing can enhance returns when prudently managed, yet excessive leverage exposes investors to heightened downside risk and financial distress, a fact underscored by the global financial crisis of 2008. Several authors have therefore examined optimal debt and consumption decisions in stochastic environments (Bank and Riedel [14]; Jin [15]; Liu and Jin [16]; Nkeki, [17, 18]). These works highlight that debt ratios are endogenous decision variables that interact with income risk, asset returns, and macroeconomic uncertainty.

Despite these advances, two important gaps remain in the literature. First, most stochastic volatility models focus exclusively on financial assets, overlooking non-financial assets such as housing, which constitute a significant component of household wealth, especially in emerging economies. Housing assets are characterized by distinct features – price volatility, depreciation, illiquidity, and borrowing-based financing—that fundamentally affect portfolio risk and consumption smoothing. Second, existing models rarely integrate stochastic volatility, housing investment, and endogenous debt decisions within a single unified framework.

This paper addresses these gaps by developing a continuous-time portfolio optimization model in which an investor simultaneously determines optimal investment in financial assets, optimal housing-related debt ratio, and optimal consumption plan under stochastic volatility. The financial market consists of a risk-free asset and a risky stock whose volatility follows a Heston-type square-root diffusion. In addition, the investor holds a non-financial housing asset whose price evolves stochastically and depreciates over time. Borrowing is explicitly modeled, and the debt ratio is treated as an endogenous control variable alongside portfolio allocation and consumption.

Using the dynamic programming approach, we derive the Hamilton-Jacobi-Bellman (HJB) equation associated with the investor's optimization problem and obtain explicit closed-form solutions under constant relative risk aversion (CRRA) preferences. The analytical results reveal how stochastic volatility, housing price risk, and leverage jointly shape optimal portfolio weights, debt ratios, and consumption paths. In particular, the model shows that volatility shocks affect not only financial investment decisions but also optimal leverage through their interaction with housing price uncertainty.

The contribution of this paper is threefold. First, it extends the stochastic volatility portfolio literature by incorporating housing assets and endogenous debt decisions into a unified continuous-time framework. Second, it provides explicit analytical characterizations of optimal investment, debt, and consumption policies under stochastic volatility. Third, it offers numerical illustrations that clarify the sensitivity of optimal decisions to volatility dynamics, housing risk, and investor risk aversion.

2. MODEL SETUP AND METHODOLOGY

This study considers a continuous-time economy over a finite horizon $[0,T]$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ satisfying the usual conditions. All stochastic processes are adapted to $\{\mathcal{F}_t\}_{t\in[0,T]}$, and trading occurs continuously without transaction costs or taxes. The market is driven by three Brownian motions W_t^S, W_t^V, and W_t^H, where:

- W_t^S drives stock price uncertainty,
- W_t^V drives volatility fluctuations under the Heston specification,
- W_t^H drives the housing price process,

with instantaneous correlations

$$dW_t^S dW_t^V = \rho\, dt, \quad dW_t^S dW_t^H = 0, \quad dW_t^V dW_t^H = 0$$

The investor allocates wealth across a risk–free asset, a risky financial asset, and a non-financial housing asset, while consuming continuously and managing debt through time.

2.1 The Financial Assets of the Investor

The investor allocates wealth to a risk-free money market account and a risky stock.

2.1.1 Risk-Free Asset

The bank account accrues interest at a constant rate $r > 0$, at any time, t, following the ordinary differential equation as suggested by Samuelson [23]:

$$dB(t) = rB(t)dt, \quad B(0) = B_0 > 0, \qquad (1)$$

2.1.2 Risky Financial Asset (Stock) with Stochastic

Let $S(t)$ denote the stock price. Its dynamics follow a Heston-type stochastic volatility specification:

$$\frac{dS(t)}{S(t)} = \mu_S dt + \sigma_S \sqrt{V(t)} dW_t^S, \qquad S(0) = S_0 > 0, \qquad (2)$$

where $\mu_s > 0$ is the expected stock return and σ_S is the volatility scaling parameter. The instantaneous variance $V(t)$ evolves according to the CIR/Heston square root diffusion [used by Cox *et al*. [20]:

$$dV(t) = k\big(\theta - V(t)\big)dt + \sigma_V \rho dW_t^V + \sigma_V \sqrt{1-\rho^2} dW_t^S, \qquad (3)$$

where $\kappa > 0$ determines the speed of mean reversion, $\theta > 0$ is the long-run volatility level, $\sigma_V > 0$ is the volatility of variance, and $\rho \in (-1, 1)$ measures the leverage effect between price and volatility shocks.

Equation (3) ensures positivity of volatility under the classical Feller condition.

2.2 Housing Asset

Let $H(t)$ denote the monetary value of the investor's housing holdings. Housing evolves as a risky non-tradable asset whose price is influenced by macroeconomic conditions:

$$\frac{dH(t)}{H(t)} = \mu_H dt + \sigma_H dW_t^H, \qquad H(0) = H_0, \qquad (4)$$

where μ_H is the expected appreciation rate and $\sigma_H > 0$ is housing price volatility. The Brownian motion W_t^H is independent of W_t^S and W_t^V.

Housing experience physical deterioration over time. Let $\delta_H > 0$ denote the depreciation rate, then (4) becomes:

$$\frac{dH(t)}{H(t)} = (\mu_H - \delta_H)dt + \sigma_H dW_t^H, \quad (5)$$

2.3 Total Assets, Leverage, and Portfolio Weights

Let $F(t)$ be the investor's total financial–market investment. A proportion $\Delta_S(t) \in [0,1]$ is allocated to stocks, and $1 - \Delta_S(t)$ to the risk-free asset:

$$\Delta_S(t)F(t) \text{ in stocks, } \quad \big(1 - \Delta_S(t)\big)F(t) \text{ in the bank account.}$$

The financial–asset dynamics are therefore:

$$dF(t) = [rF(t) + \Delta_S(t)F(t)(\mu_S - r)]dt + \Delta_S(t)F(t)\sigma_S\sqrt{V(t)}dW_t^S, \quad (6)$$

The total wealth allocated to real and financial assets is then:

$$G(t) = F(t) + H(t), \quad (7)$$

It then follows that:

$$dG(t) = [rF(t) + \Delta_S(t)F(t)(\mu_S - r) + H(t)(\mu_H - \delta_H)]dt + \Delta_S(t)F(t)\sigma_S\sqrt{V(t)}dW_t^S + \sigma_H H(t)dW_t^H, \quad (8)$$

Investors usually borrow money for investments with the hope of repaying the loan from the returns on their investment. We suppose the investor borrows a certain amount of money $L(t)$ with an interest rate $r_L(t)$ at time t to finance his investments.

Let $X(t)$ be the investor's net wealth:

$$X(t) = G(t) - L(t), \;\; X(0) = X_0 > 0, \quad (9)$$

Economically, leverage is often measured by the debt-to-asset ratio

$$\ell(t) = \frac{L(t)}{G(t)} \in [0,1), \quad (10)$$

however, for analytical convenience we work with the debt-to-wealth ratio:

$$\tilde{a}(t) = \frac{L(t)}{X(t)} \in [0, \bar{a}), \quad \bar{a} < \infty \quad (11)$$

(11) and (12) are linked by

$$\tilde{a}(t) = \frac{\ell(t)}{1-\ell(t)}, \; \ell(t) = \frac{\tilde{a}(t)}{1+\tilde{a}(t)}, \quad (12)$$

The investor allocates net wealth $X(t)$ across the financial and housing markets as follows:

- $p(t)$ is the proportion of net wealth held in financial assets, $F(t) = p(t)X(t)$. . . . . (13)
- $\pi(t)$ is the proportion of net wealth invested in the risky stock, so the amount in stock is $\pi(t)X(t)$ and the amount in the bank is $[p(t) - \pi(t)]X(t)$. . . . . (14)
- The remaining share of net wealth plus borrowed funds is invested in housing: $H(t) = \big(1 + \tilde{a}(t) - p(t)\big)X(t)$. . . . . (15)

From (13) and (15), it follows that

$$G(t) = \big(1 + \tilde{a}(t)\big)X(t) \quad (16)$$

2.4 Income Growth Dynamics

The investor earns income at the rate $\gamma(t)$, modelled as a diffusion process capturing macroeconomic and labour-market uncertainty, as identified by Jin [15]:

$$d\gamma(t) = \big[a\big(\gamma(t)\big) + \gamma(t)\eta(\omega)\big]dt + \sigma_\gamma(t)dW_t^\gamma, \quad \gamma(0) = \gamma_0 \quad (17)$$

where $a(\cdot)$ is the expected drift of the income growth rate, $\eta(\omega)$ captures the impact of the unemployment rate ω on income growth, $\sigma_\gamma > 0$ is the volatility of the income growth rate, and W_t^γ is the Brownian motion of the income growth rate.

Let $J(t)$ be the income process of the investor, defined as the product of the income growth rate $\gamma(t)$ and total asset value $G(t)$ at time t, then the change in income is given by

$$dJ(t) = \gamma(t)G(t)dt \quad (18)$$

(18) gives the income that accrues to the investor from investing the amount $G(t)$ in both financial and housing markets.

2.5 Consumption and Debt Dynamics

The investor consumes continuously from net wealth at rate $c(t) \geq 0$. Let $C(t)$ be the consumption of the investor. Then, the consumption process is given by

$$dC(t) = c(t)X(t)dt \quad (19)$$

We model the change in debt as the difference between expenditure (interest on debt and consumption) and income. Therefore, the debt dynamics is given by

$$dL(t) = r_L(t)L(t)dt + c(t)X(t)dt - \gamma(t)G(t)dt, \quad (20)$$

where $r_L(t)L(t)dt$ is interest paid on the outstanding loan, $c(t)X(t)dt$ is the consumption expenditure, and $\gamma(t)G(t)dt$ is the income inflow.

2.6 The Wealth Dynamics

The wealth process of the investor at time t is defined in (9) as the difference between the total asset value $G(t)$ and debt $L(t)$.

Using (13), (14), and (15) in equation (8), gives

$$dG(t) = \big[r(t)\big(p(t) - \pi(t)\big)X(t) + \mu_S(t)\pi(t)X(t) + (\mu_H(t) - \delta_H)\big(1 + \tilde{a}(t) - p(t)\big)X(t)\big]dt + \sigma_S\sqrt{V(t)}\pi(t)X(t)dW_t^S + \sigma_H\big(1 + \tilde{a}(t) - p(t)\big)X(t)dW_t^H \quad (21)$$

Using (11) and (16) in (20), gives

$$dL(t) = r_L(t)\tilde{a}(t)X(t)dt + c(t)X(t)dt - \gamma(t)\big(1 + \tilde{a}(t)\big)X(t)dt \quad (22)$$

From (9), we have

$$dX(t) = dG(t) - dL(t), \quad (23)$$

it then follows that

$$dX(t) = \big[\mu_H(t) - \delta_H + \gamma(t) - c(t) + (r(t) - \mu_H(t) + \delta_H)p(t) + \big(\mu_S(t) - r(t)\big)\pi(t) + \big(\mu_H(t) - \delta_H - r_L(t) + \gamma(t)\big)\tilde{a}(t)\big]X(t)dt + \pi(t)\sigma_s\sqrt{V(t)}X(t)dW_t^S + \sigma_h\big(1 + \tilde{a}(t) - p(t)\big)X(t)dW_t^H, \qquad X(0) = X_0 > 0 \quad (24)$$

Equation (24) gives the dynamics of the nominal wealth of the investor at time t.

2.7 Control Variables and Admissible Strategies

The investor exercises direct control over four dimensions simultaneously: (i) the proportion of wealth invested in risky stock assets, $\pi(t)$; (ii) the consumption rate, $c(t)$; (iii) the debt ratio, $\tilde{a}(t)$; and, (iv) the share of wealth held in the financial market, $p(t)$.

Let $X(t)$ denote total wealth, $\gamma(t)$ the stochastic income growth rate, and $V(t)$ the Heston variance process. The investor allocates a proportion $p(t) \in [0,1]$ of total wealth to the financial market and a proportion $1 - p(t)$ to housing assets. The investor allocates $\pi(t) \in [0, p(t)]$ in the risky stock and $p(t) - \pi(t)$ in the risk-free asset. Debt evolves as a proportion of total wealth, $\tilde{a}(t)X(t)$, and consumption occurs continuously at the rate $c(t)X(t)$. Altogether, the investor's decision vector is:

$$u(t) = \big(\pi(t), p(t), a(t), c(t)\big) \quad (25)$$

A control quadruple $(\pi, p, \tilde{a}, c)$ is admissible if it is progressively measurable, square-integrable, and satisfies:

$$0 \leq \pi(t) \leq p(t) \leq 1, \qquad 0 \leq c(t) < \infty, \qquad \tilde{a}(t) \geq 0, \quad (26)$$

with wealth bounded away from bankruptcy almost surely. This ensures the investor cannot borrow or consume beyond admissibility tolerances and cannot hold negative proportions of housing or financial assets.

Let $\mathcal{A}$ be the collection of all admissible strategies, we have that $\mathcal{A}$ can be defined as

$$\mathcal{A} = \Big\{u(t) = \{\pi(t), p(t), \tilde{a}(t), c(t)\big) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} : \ \mathbb{E}\int_0^T \pi(t)^2 dt < \infty;\ \mathbb{E}\int_0^T p(t)^2 dt < \infty;\ \mathbb{E}\int_0^T \tilde{a}(t)^2 dt < \infty;\ \mathbb{E}\int_0^T c(t)^2 dt < \infty\Big\} \ldots\ldots (27)$$

2.8 Optimization Problem

The desire of the investor is to choose investment policies for risky financial and non-financial assets, debt ratio, and consumption plan that will maximize the total expected discounted utility of the intermediate consumption and terminal wealth.

Setting the wealth dynamics of the investor, $X(t)$ as x and the income growth rate $\gamma(t)$ as γ, for CRRA utility:

$$B(C) = \frac{C^{1-\alpha}}{1-\alpha},\ \ \alpha < 0, \alpha \neq 1, \ldots\ldots (28)$$

the optimization objective function is given by:

$$F(t, x, \gamma, V) \coloneqq \sup_{u \in \mathcal{A}} \mathbb{E}\left[\int_0^T e^{-\beta t} B(C) dt + e^{-\beta T} B\big(X(T)\big)\right], \ldots\ldots (29)$$

where $\beta > 0$ is the subjective discount rate, $C = c(t)X(t)$ is the consumption of the investor of which process is defined in (19), and $B(\cdot)$ is a CRRA (power) utility function.

2.9 Solution Methodology

The optimization problem is solved using the dynamic programming principle. The associated Hamilton-Jacobi-Bellman (HJB) equation is derived with state variables $\big(X(t), V(t), \gamma(t)\big)$. Closed-form solutions for the optimal portfolio allocation, debt ratio, and consumption policy are obtained under power utility assumptions.

3. HJB FORMULATION AND OPTIMAL CONTROLS

3.1 Hamilton-Jacobi-Bellman Equation

Applying dynamic programming principle, and considering the three state variables x (24), V (3), and γ (17), produces the four-control Hamilton-Jacobi-Bellman (HJB) equation:

$$0 = F_t + \max_{\pi, p, \tilde{a}, c}\left[J^{x,\gamma,V} F + e^{-\delta t} B(C)\right], \ldots\ldots (30)$$

where $J^{x,\gamma,V}$ is the infinitesimal generator of the wealth-income-volatility system

Expanding terms, wealth contributions enter as:

$$J^x F = [\mu_H - \delta_H + \gamma - c + rp + \pi(\mu_S - r) + \mu_H(\tilde{a} - p) + \delta_H(p - \tilde{a}) - (r_L - \gamma)\tilde{a}]xF_x + \frac{1}{2}x^2\pi^2\sigma_S^2 V F_{xx} + \frac{1}{2}x^2\sigma_H^2 F_{xx} + \frac{1}{2}x^2\sigma_H^2\tilde{a}^2 F_{xx} - \frac{1}{2}x^2\sigma_H^2 p^2 F_{xx}, \ldots\ldots (31)$$

income contributions enter as:

$$J^\gamma F = \gamma(a + \eta)F_\gamma + \frac{1}{2}\sigma_\gamma^2 F_{\gamma\gamma}, \ldots\ldots (32)$$

volatility contributions enter as:

$$J^V F = k(\theta - V)F_V + \frac{1}{2}\sigma_V^2 F_{VV} \ldots\ldots (33)$$

Cross-terms in $F_{x\gamma}$, F_{xV}, and $F_{\gamma V}$ emerge directly from stochastic co-movement between wealth, volatility, and income shocks, as follows:

$$J^{x\gamma} F = \gamma(a + \eta)[\mu_H - \delta_H + \gamma - c + rp + \pi(\mu_S - r) + \mu_H(\tilde{a} - p) + \delta_H(p - \tilde{a}) - (r_L - \gamma)\tilde{a}]x\rho_{\gamma x}F_{\gamma x} + \gamma(a + \eta)\pi\sigma_S\sqrt{V}x\rho_{\gamma x}F_{\gamma x} + \gamma(a + \eta)\sigma_H x\rho_{\gamma x}F_{\gamma x} + \gamma(a + \eta)\sigma_H\tilde{a}x\rho_{\gamma x}F_{\gamma x} - \gamma(a + \eta)\sigma_H p x\rho_{\gamma x}F_{\gamma x} + \sigma_\gamma[\mu_H - \delta_H + \gamma - c + rp + \pi(\mu_S - r) + \mu_H(\tilde{a} - p) + \delta_H(p - \tilde{a}) - (r_L - \gamma)\tilde{a}]x\rho_{\gamma x}F_{\gamma x} + \pi\sigma_\gamma\sigma_S\sqrt{V}x\rho_{\gamma x}F_{\gamma x} + \sigma_\gamma\sigma_H x\rho_{\gamma x}B_{\gamma x} + \sigma_\gamma\sigma_H\tilde{a}x\rho_{\gamma x}F_{\gamma x} - \sigma_\gamma\sigma_H p x\rho_{\gamma x}F_{\gamma x}, \ldots\ldots (34)$$

$J^{xV}F = k(\theta - V)[\mu_H - \delta_H + \gamma - c + rp + \pi(\mu_S - r) + \mu_H(\tilde{a} - p) + \delta_H(p - \tilde{a}) - (r_L - \gamma)\tilde{a}]x\rho_{Vx}F_{Vx} + k(\theta - V)\pi\sigma_S\sqrt{V}x\rho_{Vx}F_{Vx} + k(\theta - V)\sigma_H x\rho_{Vx}F_{Vx} + k(\theta - V)\sigma_H\tilde{a}x\rho_{Vx}F_{Vx} - k(\theta - V)\sigma_H px\rho_{Vx}F_{Vx} + \sigma_V\rho[\mu_H - \delta_H + \gamma - c + rp + \pi(\mu_S - r) + \mu_H(\tilde{a} - p) + \delta_H(p - \tilde{a}) - (r_L - \gamma)\tilde{a}]x\rho_{Vx}F_{Vx} + \sigma_V\rho\pi\sigma_S\sqrt{V}x\rho_{Vx}F_{Vx} + \sigma_V\rho\sigma_H x\rho_{Vx}F_{Vx} + \sigma_V\rho\sigma_H\tilde{a}x\rho_{Vx}F_{Vx} - \sigma_V\rho\sigma_H px\rho_{Vx}F_{Vx} + \sigma_V\sqrt{1-\rho^2}[\mu_H - \delta_H + \gamma - c + rp + \pi(\mu_S - r) + \mu_H(\tilde{a} - p) + \delta_H(p - \tilde{a}) - (r_L - \gamma)\tilde{a}]x\rho_{Vx}F_{Vx} + \sigma_V\sqrt{1-\rho^2}\pi\sigma_S\sqrt{V}x\rho_{Vx}F_{Vx} + \sigma_V\sqrt{1-\rho^2}\sigma_H x\rho_{Vx}F_{Vx} + \sigma_V\sqrt{1-\rho^2}\sigma_H\tilde{a}x\rho_{Vx}F_{Vx} + \sigma_V\sqrt{1-\rho^2}\sigma_H px\rho_{Vx}F_{Vx}, \ldots \quad (35)$

$J^{\gamma V}F = \gamma k(a + \eta)(\theta - V)\rho_{\gamma V}F_{\gamma V} + \gamma(a + \eta)\sigma_V\rho\rho_{\gamma V}F_{\gamma V} + \gamma(a + \eta)\sigma_V\sqrt{1-\rho^2}\rho_{\gamma V}F_{\gamma V} + k(\theta - V)\sigma_\gamma\rho_{\gamma V}F_{\gamma V} + \sigma_\gamma\sigma_V\rho\rho_{\gamma V}F_{\gamma V} + \sigma_\gamma\sigma_V\sqrt{1-\rho^2}\rho_{\gamma V}F_{\gamma V} \ldots \quad (36)$

where $B_t, B_x, B_{xx}, B_\gamma, B_{\gamma\gamma}, B_V, B_{VV}, B_{\gamma x}, B_{Vx}$, and $B_{\gamma V}$ denote partial derivatives of first-order and second-order with respect to time t, wealth process x, income growth rate γ, and volatility process V.

Because $F(\cdot)$ and $B(\cdot)$ are concave and the control set in $(p, \pi, \tilde{a}, c)$ is convex, the Hamiltonian in (30) is concave in the control variables, which guarantees that the first–order conditions indeed characterize the optimal controls.

Explicitly, equation (3) is:

$0 = F_t + J^xF + J^\gamma F + J^VF + J^{x\gamma}F + J^{xV}F + J^{\gamma V}F + e^{-\beta t}B(C) \ldots \quad (37)$

To determine the optimal portfolios, consumption plan, and debt ratio, we have to specify the forms of the utility functions B.

3.2 Power Utility

To obtain explicit optimal policies we specialize to a constant-relative-risk-aversion (CRRA) specification for intermediate consumption (28), and search for a value function of the isoelastic form

$F(x, \gamma, V, t) = \frac{(x+V)^{1-\varepsilon}e^{g(t,\gamma)}}{1-\varepsilon}, \ldots \quad (38)$

where $\varepsilon \in (0,1) \cup (1,\infty)$ is the risk aversion factor in relation to the wealth process x, γ is the income-growth state, V is the variance process, and g is a deterministic function to be determined. Under (38) the first- and second- order derivatives of F are proportional to F itself:

$F_t = g_tF, \quad F_x = \frac{(1-\varepsilon)F}{x+V}, \quad F_{xx} = \frac{-\varepsilon(1-\varepsilon)F}{(x+V)^2}, \quad F_V = \frac{(1-\varepsilon)F}{x+V}, \quad F_{VV} = \frac{-\varepsilon(1-\varepsilon)F}{(x+V)^2}, \quad F_\gamma = g_\gamma F,$

$F_{\gamma\gamma} = (g_\gamma^2 + g_{\gamma\gamma})F, \quad F_{\gamma x} = \frac{(1-\varepsilon)g_\gamma F}{x+V}, \quad F_{\gamma V} = \frac{(1-\varepsilon)g_\gamma F}{x+V}, \quad F_{Vx} = \frac{-\varepsilon(1-\varepsilon)F}{(x+V)^2} \ldots (39)$

Substituting these (39) into (37) and collecting like terms yields

$0 = \max_{\pi,p,\tilde{a},c}\Big\{g_tF + \frac{(1-\varepsilon)}{x+V}\Phi_0 xF - \frac{\varepsilon(1-\varepsilon)}{2(x+V)^2}[\sigma_H^2 + \sigma_S^2V\pi^2 - \sigma_H^2p^2 + \sigma_H^2\tilde{a}^2]x^2F + \gamma(a + \eta)g_\gamma F + \frac{1}{2}\sigma_\gamma^2(g_\gamma^2 + g_{\gamma\gamma})F + \frac{k(\theta-V)(1-\varepsilon)}{x+V}F - \frac{\sigma_V^2\varepsilon(1-\varepsilon)}{2(x+V)^2}F + \frac{(1-\varepsilon)g_\gamma}{x+V}\Phi_0 x\gamma(a + \eta)\rho_{\gamma x}F + \frac{(1-\varepsilon)g_\gamma}{x+V}\Phi_1 x\gamma(a + \eta)\rho_{\gamma x}F + \frac{(1-\varepsilon)g_\gamma}{x+V}\Phi_0 x\sigma_\gamma\rho_{\gamma x}F + \frac{(1-\varepsilon)g_\gamma}{x+V}\Phi_1 x\sigma_\gamma\rho_{\gamma x}F - \frac{\varepsilon(1-\varepsilon)}{(x+V)^2}\Phi_0 xk(\theta - V)\rho_{Vx}F - \frac{\varepsilon(1-\varepsilon)}{(x+V)^2}\Phi_1 xk(\theta - V)\rho_{Vx}F - \frac{\varepsilon(1-\varepsilon)}{(x+V)^2}\Phi_0 x\sigma_V\rho\rho_{Vx}F - \frac{\varepsilon(1-\varepsilon)}{(x+V)^2}\Phi_1 x\sigma_V\rho\rho_{Vx}F - \frac{\varepsilon(1-\varepsilon)}{(x+V)^2}\Phi_0 x\sigma_V\sqrt{1-\rho^2}\rho_{Vx}F - \frac{\varepsilon(1-\varepsilon)}{(x+V)^2}\Phi_1 x\sigma_V\sqrt{1-\rho^2}\rho_{Vx}F + \frac{\gamma k(a+\eta)(\theta-V)(1-\varepsilon)g_\gamma\rho_{\gamma V}}{x+V}F + \frac{\gamma(a+\eta)\sigma_V\rho(1-\varepsilon)g_\gamma\rho_{\gamma V}}{x+V}F + \frac{\gamma(a+\eta)\sigma_V\sqrt{1-\rho^2}(1-\varepsilon)g_\gamma\rho_{\gamma V}}{x+V}F + \frac{k(\theta-V)\sigma_\gamma(1-\varepsilon)g_\gamma\rho_{\gamma V}}{x+V}F + \frac{\sigma_\gamma\sigma_V\rho(1-\varepsilon)g_\gamma\rho_{\gamma V}}{x+V}F + \frac{\sigma_\gamma\sigma_V\sqrt{1-\rho^2}(1-\varepsilon)g_\gamma\rho_{\gamma V}}{x+V}F + e^{-\beta t}\frac{C^{1-\alpha}}{1-\alpha}\Big\} \ldots \quad (40)$

where $\Phi_0 = (\mu_H - \delta_H + \gamma) + (\mu_S - r)\pi + (r - \mu_H + \delta_H)p + (\mu_H - \delta_H - r_L + \gamma)\tilde{a} - c$ and $\Phi_1 = \sigma_H + \sigma_S\sqrt{V}\pi + \sigma_H p + \sigma_H \tilde{a}$

Dividing (40) through by $-(1-\varepsilon)F < 0$, we obtain the equivalent minimization problem. Thus, we have

$$0 = \min_{\pi,p,\tilde{a},c}\left\{-\frac{g_t}{1-\varepsilon} - \frac{\Phi_0 x}{x+V} + \frac{\varepsilon x^2}{2(x+V)^2}[\sigma_H^2 + \sigma_S^2 V\pi^2 - \sigma_H^2 p^2 + \sigma_H^2\tilde{a}^2] - \frac{\gamma(a+\eta)}{1-\varepsilon}g_\gamma - \frac{\sigma_\gamma^2}{2(1-\varepsilon)}\left(g_\gamma^2 + g_{\gamma\gamma}\right) - \frac{k(\theta-V)}{x+V} + \frac{\sigma_V^2\varepsilon}{2(x+V)^2} - \frac{g_\gamma}{x+V}\Phi_0 x\gamma(a+\eta)\rho_{\gamma x} - \frac{g_\gamma}{x+V}\Phi_1 x\gamma(a+\eta)\rho_{\gamma x} - \frac{g_\gamma}{x+V}\Phi_0 x\sigma_\gamma\rho_{\gamma x} - \frac{g_\gamma}{x+V}\Phi_1 x\sigma_\gamma\rho_{\gamma x} + \frac{\varepsilon}{(x+V)^2}\Phi_0 xk(\theta-V)\rho_{Vx} + \frac{\varepsilon}{(x+V)^2}\Phi_1 xk(\theta-V)\rho_{Vx} + \frac{\varepsilon}{(x+V)^2}\Phi_0 x\sigma_V\rho\rho_{Vx} + \frac{\varepsilon}{(x+V)^2}\Phi_1 x\sigma_V\rho\rho_{Vx} + \frac{\varepsilon}{(x+V)^2}\Phi_0 x\sigma_V\sqrt{1-\rho^2}\rho_{Vx} + \frac{\varepsilon}{(x+V)^2}\Phi_1 x\sigma_V\sqrt{1-\rho^2}\rho_{Vx} - \frac{\gamma k(a+\eta)(\theta-V)g_\gamma\rho_{\gamma V}}{x+V} - \frac{\gamma(a+\eta)\sigma_V\rho g_\gamma\rho_{\gamma V}}{x+V} - \frac{\gamma(a+\eta)\sigma_V\sqrt{1-\rho^2}g_\gamma\rho_{\gamma V}}{x+V} - \frac{k(\theta-V)\sigma_\gamma g_\gamma\rho_{\gamma V}}{x+V} - \frac{\sigma_\gamma\sigma_V\rho g_\gamma\rho_{\gamma V}}{x+V} - \frac{\sigma_\gamma\sigma_V\sqrt{1-\rho^2}g_\gamma\rho_{\gamma V}}{x+V} - \frac{e^{-\beta t}C^{1-\alpha}}{(1-\alpha)(x+V)^{1-\varepsilon}e^{g(t,\gamma)}}\right\} \quad (41)$$

The minimization problem (41) is quadratic in the portfolio and debt controls $(\pi, p, \tilde{a})$ and strictly concave in c. The first-order conditions therefore yield closed-form expressions for the optimal policies $(\pi^*, p^*, \tilde{a}^*, c^*)$ as affine functions of the state variables (x, γ, V).

3.3 Optimal Debt Ratio

We consider the optimal debt ratio of the investor at time t.

The optimal policy for the debt ratio $\tilde{a}^*$ of the investor at time t, is defined as

$$\tilde{a}^* = argmin_{\tilde{a}} f_1(\tilde{a}), \quad (42)$$

where

$$f_1(\tilde{a}) = -\frac{(\mu_H-\delta_H-r_L+\gamma)\tilde{a}x}{x+V} + \frac{\varepsilon\sigma_H^2\tilde{a}^2x^2}{2(x+V)^2} - \left[\frac{\gamma(a+\eta)(\mu_H-\delta_H-r_L+\gamma)\tilde{a}xg_\gamma}{x+V} + \frac{\gamma(a+\eta)\sigma_H\tilde{a}xg_\gamma}{x+V} + \frac{\sigma_\gamma(\mu_H-\delta_H-r_L+\gamma)\tilde{a}xg_\gamma}{x+V} + \frac{\sigma_\gamma\sigma_H\tilde{a}xg_\gamma}{x+V}\right]\rho_{\gamma x} + \left[\frac{k(\theta-V)(\mu_H-\delta_H-r_L+\gamma)\tilde{a}\varepsilon x}{(x+V)^2} + \frac{k(\theta-V)\sigma_H\tilde{a}\varepsilon x}{(x+V)^2} + \frac{\sigma_V\rho(\mu_H-\delta_H-r_L+\gamma)\tilde{a}\varepsilon x}{(x+V)^2} + \frac{\sigma_V\rho\sigma_H\tilde{a}\varepsilon x}{(x+V)^2} + \frac{\sigma_V\sqrt{1-\rho^2}(\mu_H-\delta_H-r_L+\gamma)\tilde{a}\varepsilon x}{(x+V)^2} + \frac{\sigma_V\sqrt{1-\rho^2}\sigma_H\tilde{a}\varepsilon x}{(x+V)^2}\right]\rho_{Vx} \quad (43)$$

Let $\tilde{a}^*$ be the optimal debt ratio of the investor at time t, then by first-order conditions

$$\tilde{a}^* = \frac{(x+V)(\mu_H-\delta_H-r_L+\gamma)}{x\sigma_H^2\varepsilon}(1+\lambda) + \frac{(x+V)\lambda}{x\sigma_H\varepsilon} - \frac{(\mu_H-\delta_H-r_L+\gamma)\tau}{x\sigma_H^2} - \frac{\tau}{x\sigma_H} \quad (44)$$

where

$$\lambda = (\gamma a + \gamma\eta + \sigma_\gamma)g_\gamma\rho_{\gamma x} \text{ and } \tau = \left(k(\theta-V) + \sigma_V\rho + \sigma_V\sqrt{1-\rho^2}\right)\rho_{Vx} \quad (45)$$

The above optimal debt ratio can also be written as

$$\tilde{a}^* = \underbrace{\frac{(x+V)(\mu_H-\delta_H-r_L+\gamma)}{x\sigma_H^2\varepsilon}}_{\varphi_1} + \underbrace{\frac{(x+V)(\mu_H-\delta_H-r_L+\gamma)\lambda}{x\sigma_H^2\varepsilon}}_{\varphi_2} + \underbrace{\frac{(x+V)\lambda}{x\sigma_H\varepsilon}}_{\varphi_3} - \underbrace{\frac{(\mu_H-\delta_H-r_L+\gamma)\tau}{x\sigma_H^2}}_{\varphi_4} - \underbrace{\frac{\tau}{x\sigma_H}}_{\varphi_5} \quad (46)$$

3.4 Optimal Investment Policy

3.4.1 Optimal Allocation to the Financial Market

The optimal policy for the portfolio weights strategy in financial market assets p^* of the investor at time t, is defined as

$$p^* = argmin_\pi f_2(p), \quad (47)$$

where

$$f_2(p) = -\frac{(r-\mu_H+\delta_H)px}{x+V} + \frac{\varepsilon\sigma_H^2 p^2 x^2}{2(x+V)^2} - \left[\frac{\gamma(a+\eta)(r-\mu_H+\delta_H)pxg_\gamma}{x+V} + \frac{\gamma(a+\eta)\sigma_H pxg_\gamma}{x+V} + \frac{\sigma_\gamma(r-\mu_H+\delta_H)pxg_\gamma}{x+V} + \frac{\sigma_\gamma\sigma_H pxg_\gamma}{x+V}\right]\rho_{\gamma x} + \left[\frac{k(\theta-V)(r-\mu_H+\delta_H)p\varepsilon x}{(x+V)^2} + \frac{k(\theta-V)\sigma_H p\varepsilon x}{(x+V)^2} + \frac{\sigma_V\rho(r-\mu_H+\delta_H)p\varepsilon x}{(x+V)^2} + \frac{\sigma_V\rho\sigma_H p\varepsilon x}{(x+V)^2} + \frac{\sigma_V\sqrt{1-\rho^2}(r-\mu_H+\delta_H)p\varepsilon x}{(x+V)^2} + \frac{\sigma_V\sqrt{1-\rho^2}\sigma_H p\varepsilon x}{(x+V)^2}\right]\rho_{Vx} \quad (48)$$

Let p^* be the optimal investment policy strategy in financial market assets of the investor at time t, then by first-order conditions

$$p^* = \frac{(x+V)(r-\mu_H+\delta_H)}{x\sigma_H^2\varepsilon}(1+\lambda) + \frac{(x+V)\lambda}{x\sigma_H\varepsilon} - \frac{(r-\mu_H+\delta_H)\tau}{x\sigma_H^2} - \frac{\tau}{x\sigma_H} \quad (49)$$

The above optimal investment strategy can also be written as

$$p^* = \underbrace{\frac{(x+V)(r-\mu_H+\delta_H)}{x\sigma_H^2\varepsilon}}_{\vartheta_1} + \underbrace{\frac{(x+V)(r-\mu_H+\delta_H)\lambda}{x\sigma_H^2\varepsilon}}_{\vartheta_2} + \underbrace{\frac{(x+V)\lambda}{x\sigma_H\varepsilon}}_{\vartheta_3} - \underbrace{\frac{(r-\mu_H+\delta_H)\tau}{x\sigma_H^2}}_{\vartheta_4} - \underbrace{\frac{\tau}{x\sigma_H}}_{\vartheta_5} \quad (50)$$

3.4.2 Optimal Allocation to Risky Asset

The optimal policy for the portfolio weights strategy in risky assets π^* of the investor at time t, is defined as

$$\pi^* = argmin_\pi f_3(\pi), \quad (51)$$

where

$$f_3(\pi) = -\frac{(\mu_S-r)\pi x}{x+V} + \frac{\varepsilon\sigma_S^2 V\pi^2 x^2}{2(x+V)^2} - \left[\frac{\gamma(a+\eta)(\mu_S-r)\pi xg_\gamma}{x+V} + \frac{\gamma(a+\eta)\sigma_S\sqrt{V}\pi xg_\gamma}{x+V} + \frac{\sigma_\gamma(\mu_S-r)\pi xg_\gamma}{x+V} + \frac{\sigma_\gamma\sigma_S\sqrt{V}\pi xg_\gamma}{x+V}\right]\rho_{\gamma x} + \left[\frac{k(\theta-V)(\mu_S-r)\pi\varepsilon x}{(x+V)^2} + \frac{k(\theta-V)\sigma_S\sqrt{V}\pi\varepsilon x}{(x+V)^2} + \frac{\sigma_V\rho(\mu_S-r)\pi\varepsilon x}{(x+V)^2} + \frac{\sigma_V\rho\sigma_S\sqrt{V}\pi\varepsilon x}{(x+V)^2} + \frac{\sigma_V\sqrt{1-\rho^2}(\mu_S-r)\pi\varepsilon x}{(x+V)^2} + \frac{\sigma_V\sqrt{1-\rho^2}\sigma_S\sqrt{V}\pi\varepsilon x}{(x+V)^2}\right]\rho_{Vx} \quad (52)$$

Let π^* be the optimal investment policy strategy in risky assets of the investor at time t, then by first-order conditions

$$\pi^* = \frac{(x+V)(\mu_S-r)}{x\sigma_S^2 V\varepsilon}(1+\lambda) + \frac{(x+V)\lambda}{x\sigma_S\sqrt{V}\varepsilon} - \frac{(\mu_S-r)\tau}{x\sigma_S^2 V} - \frac{\tau}{x\sigma_S\sqrt{V}} \quad (53)$$

The above optimal investment strategy can also be written as

$$\pi^* = \underbrace{\frac{(x+V)(\mu_S-r)}{x\sigma_S^2 V\varepsilon}}_{\zeta_1} + \underbrace{\frac{(x+V)(\mu_S-r)\lambda}{x\sigma_S^2 V\varepsilon}}_{\zeta_2} + \underbrace{\frac{(x+V)\lambda}{x\sigma_S\sqrt{V}\varepsilon}}_{\zeta_3} - \underbrace{\frac{(\mu_S-r)\tau}{x\sigma_S^2 V}}_{\zeta_4} - \underbrace{\frac{\tau}{x\sigma_S\sqrt{V}}}_{\zeta_5} \quad (54)$$

3.5 Optimal Consumption Policy

Let c^* be the optimal consumption strategy of the investor at time t, then by first-order conditions

$$c^* = Y^{-\frac{1}{\alpha}}(x+V)^{\frac{\varepsilon-1}{\alpha}}e^{-\left(\frac{\beta t+g}{\alpha}\right)} \quad (55)$$

where

$$Y = \frac{x}{x+V}\left(1+\lambda-\frac{\tau\varepsilon}{x+V}\right),\ \lambda = (\gamma a+\gamma\eta+\sigma_\gamma)g_\gamma\rho_{\gamma x},\ \text{and}\ \tau = \left(k(\theta-V)+\sigma_V\rho+\sigma_V\sqrt{1-\rho^2}\right)\rho_{Vx} \quad (56)$$

3.6 The Explicit Form of the HJB Equation

The explicit form of the HJB equation (41) is given by

$$-\frac{g_t}{1-\varepsilon} - \frac{x}{x+V}\left(\mu_H-\delta_H+\gamma-Y^{-\frac{1}{\alpha}}(x+V)^{\frac{\varepsilon-1}{\alpha}}e^{-\left(\frac{\beta t+g}{\alpha}\right)}\right)\left(1+\lambda-\frac{\tau\varepsilon}{x+V}\right) - \frac{(\mu_S-r)^2(1+\lambda)^2}{2\sigma_S^2 V\varepsilon} + \frac{(\mu_S-r)^2(1+\lambda)\tau}{(x+V)\sigma_S^2 V} - \frac{(\mu_s-r)^2\tau^2\varepsilon}{2(x+V)^2\sigma_S^2 V} - \frac{(r-\mu_H+\delta_H)^2(1+\lambda)^2}{2\sigma_H^2\varepsilon} + \frac{(r-\mu_H+\delta_H)^2(1+\lambda)\tau}{(x+V)\sigma_H^2} - \frac{(r-\mu_H+\delta_H)^2\tau^2\varepsilon}{2(x+V)^2\sigma_H^2} - \frac{(\mu_H-\delta_H-r_L+\gamma)^2(1+\lambda)^2}{2\sigma_H^2\varepsilon} + \frac{(\mu_H-\delta_H-r_L+\gamma)^2(1+\lambda)\tau}{(x+V)\sigma_H^2} - \frac{(\mu_H-\delta_H-r_L+\gamma)^2\tau^2\varepsilon}{2(x+V)^2\sigma_H^2} + \frac{3\lambda^2}{2\varepsilon} + \frac{3\tau^2\varepsilon}{2(x+V)^2} - \frac{3\lambda\tau}{x+V} + \frac{x^2\sigma_H^2\varepsilon}{2(x+V)^2} - \frac{\gamma(a+\eta)g_\gamma}{1-\varepsilon} - \frac{\sigma_\gamma^2\left((g_\gamma)^2+g_{\gamma\gamma}\right)}{2(1-\varepsilon)} -$$

$\frac{k(\theta-V)}{x+V}+\frac{\sigma_V^2\varepsilon}{2(x+V)^2}-\frac{\gamma(a+\eta)k(\theta-V)g_\gamma}{x+V}\rho_{\gamma V}-\frac{\gamma(a+\eta)\sigma_V\rho g_\gamma}{x+V}\rho_{\gamma V}-\frac{\gamma(a+\eta)\sigma_V\sqrt{1-\rho^2}g_\gamma}{x+V}\rho_{\gamma V}-\frac{k(\theta-V)\sigma_\gamma g_\gamma}{x+V}\rho_{\gamma V}-$
$\frac{\sigma_\gamma\sigma_V\rho g_\gamma}{x+V}\rho_{\gamma V}-\frac{\sigma_\gamma\sigma_V\sqrt{1-\rho^2}g_\gamma}{x+V}\rho_{\gamma V}=0$ . . . . (57)

RESULTS AND DISCUSSION

3.7.1 Optimal Debt Ratio

The optimal debt ratio $\tilde{\alpha}^*$ obtained in equation (44) admits the decomposition (46), where λ and τ reflect the joint effects of income risk, housing-price risk, and volatility risk.
This structure yields five economically meaningful components, each capturing a distinct driver of leverage:

1. **Speculative debt demand $\boldsymbol{\varphi_1}$.**
   This component increases with investor wealth and volatility, and with the excess return on housing relative to loan cost. It decreases with σ_H^2. Thus, when housing-price risk is low relative to expected appreciation, the investor optimally takes on more leverage.
2. **Income-growth speculative component $\boldsymbol{\varphi_2}$.**
   Driven by $\lambda=(\gamma a+\gamma\eta+\sigma_\gamma)g_\gamma\rho_{\gamma x}$, this term increases with income growth and the correlation between income shocks and wealth. It decreases with stock and housing volatility and with higher CRRA risk aversion.
3. **Income-risk hedging component $\boldsymbol{\varphi_3}$.**
   This component hedges the covariance between income shocks and wealth. It rises with the volatility process V, income growth, and the correlation $\rho_{\gamma x}$, but falls with σ_H^2 and σ_H. Thus, households facing income volatility hedge by partially adjusting leverage.
4. **Housing-risk hedging component $\boldsymbol{\varphi_4}$.**
   This term responds to shocks in the housing-price volatility process. It increases when V is high and when income growth contributes positively to consumption and housing wealth but falls with σ_H^2. Higher housing volatility induces deleveraging.
5. **Volatility-risk hedging component $\boldsymbol{\varphi_5}$.**
   Driven by $\tau=\left(k(\theta-V)+\sigma_V\rho+\sigma_V\sqrt{1-\rho^2}\right)\rho_{Vx}$, this term hedges shocks to the stochastic volatility process. It is increasing in the volatility process V and decreasing in both wealth x and the dispersion of housing-price shocks.

Overall, the decomposition reveals a nuanced pattern: leverage increases when the expected return on housing exceeds the loan rate and when income and volatility positively co-move with wealth. However, the investor optimally reduces debt when volatility risk or housing-price risk becomes elevated. This heterogeneity in components highlights why the debt ratio is highly sensitive to both income dynamics and stochastic volatility.

3.7.2 Optimal Allocation to the Financial Market

The optimal portfolio weight p^* for investment in financial-market assets (equation 49) decomposes into five components analogous to the debt case:

1. **Speculative portfolio $\boldsymbol{\vartheta_1}$**: increases with $(x+V)$ and the spread $(r-\mu_H+\delta_H)$; decreases with σ_H^2.
2. **Income-growth speculative component $\boldsymbol{\vartheta_2}$**: increases with income growth γ, with the correlation between income risk and wealth, and with the volatility process.
3. **Income-risk hedging component $\boldsymbol{\vartheta_3}$**: lowers exposure when income-wealth covariance is high; increases with V.

4. **Volatility-risk hedging component $\boldsymbol{\vartheta_4}$**: protects against shocks in the variance process V; decreases in σ_H.
5. **Housing-risk hedging component $\boldsymbol{\vartheta_5}$**: reduces exposure when housing-price volatility or volatility-of-volatility rises.
6.

High volatility or high covariance between housing and income shocks reduces the attractiveness of financial-market investment. When the variance process V is low, the investor optimally increases exposure to financial assets, but deleverages from p as V rises.

3.7.3 Optimal Allocation to the Risky Market

The optimal risky-asset weight π^* (equation 53) shares the same five components but is driven by the excess return of the risky asset $(\mu_S - r)$ rather than housing or loan costs. The dominant mechanisms are:

1. **Speculative term $\boldsymbol{\zeta_1}$**: positively related to $(x + V)$ and $(\mu_S - r)$; decreases with σ_S^2.
2. **Income-speculative term $\boldsymbol{\zeta_2}$**: relevant when income shocks correlate with wealth; increases with V.
3. **Income-hedging term $\boldsymbol{\zeta_3}$**: reduces risky-asset exposure during periods of income instability.
4. **Volatility-hedging term $\boldsymbol{\zeta_4}$**: particularly sensitive to movements of the stochastic variance process; decreases with $\sigma_S\sqrt{V}$.
5. **Housing-hedging term $\boldsymbol{\zeta_5}$**: arises because housing acts as a non-tradeable asset correlated with wealth.

The optimal risky-asset weight is very sensitive to the volatility process. When V rises, the hedging components dominate, and the investor optimally reduces exposure to the financial risky asset, even if expected returns remain favorable. This reinforces the importance of stochastic volatility in shaping optimal portfolio choice.

3.7.4 Optimal Consumption

The optimal consumption rule c^* (equation 55) exhibits intuitive homogeneity in wealth and volatility. The term Y captures the interaction between wealth, income dynamics, and intertemporal substitution.

Consumption rises with wealth $x + V$ and falls with higher effective discounting α. As $\alpha \to \infty$, consumption tends to consume nearly all available resources; as $\alpha \to 0$, the investor prefers to allocate more wealth to investment. Increased volatility V raises consumption through precautionary motives but only proportionally to the CRRA parameter.

3.7.5 Overall Insights

Three broad qualitative conclusions emerge:

1. **Stochastic volatility plays a central role across all decisions**.
   Every optimal policy includes at least one volatility-hedging component. As V increases, the investor systematically reduces leverage, risky-asset exposure, and financial-market allocation.
2. **Housing-price volatility is a major determinant of leverage and asset allocation**.
   High σ_H reduces debt and risky-asset exposure, reflecting the non-tradable nature of housing and its effect on wealth risk.
3. **Income dynamics interact strongly with investment and debt decisions**.

Correlation between income shocks and wealth $\rho_{\gamma x}$ amplifies both speculative and hedging motives, with the sign determined by whether income risk augments or dilutes effective wealth.

4. NUMERICAL ILLUSTRATIONS AND DISCUSSION

The objective of this section is twofold. First, it provides quantitative illustrations of the analytical results derived in Sections 2 and 3, thereby clarifying the economic implications of the optimal investment, debt, consumption, and financial-market allocation policies. Second, it examines the sensitivity of these optimal policies to changes in market volatility, risk preferences, and housing price uncertainty.

To ensure economic plausibility, parameter values are informed by observed characteristics of the Nigerian financial market, with particular reference to equity price behavior. Daily closing prices of SEPLAT Petroleum Development Company Plc, listed on the Nigerian Exchange Group (NGX), were used as a representative equity series due to their relatively high liquidity and consistent trading history over the sample period. The stock price data were converted to continuously compounded returns. These returns were employed to obtain preliminary estimates of the drift and volatility parameters of the risky asset. While the analytical model assumes a Heston-type stochastic volatility process, full maximum likelihood or Bayesian estimation of the Heston parameters is beyond the scope of this paper. Instead, the empirical data are used to anchor the numerical magnitudes of the parameters, ensuring that simulations reflect realistic market conditions rather than purely arbitrary values.

The volatility process parameters were selected to satisfy standard admissibility conditions, including positivity of the variance process and mean-reversion. In particular, the Feller condition is respected to ensure that the volatility process remains strictly positive. Housing price dynamics are modeled separately and are not estimated directly from the SEPLAT data. Instead, housing appreciation and volatility parameters are chosen to reflect stylized facts from emerging housing markets, where price growth tends to be moderate, but uncertainty can be substantial. In the numerical illustrations we approximate the macro channel with a constant reduced-form coefficient η.

All model parameters are expressed in annualized terms, consistent with continuous-time finance conventions. The benchmark parameter values used in the numerical simulations are summarized below:

$r = 0.125$, $\mu_S = 0.28$, $\sigma_S = 4$, $\mu_H = 0.19$, $\sigma_H = 1$, $\delta_H = 0.02$, $r_L = 0.2$, $\gamma = 0.05$, $\sigma_\gamma = 0.155$, $k = 0.9$, $\theta = 0.14$, $\sigma_V = 0.7$, $\rho = -0.3$, $\epsilon = \{1.5, 3, 6\}$, $a = 0.1$, $\eta = 0.05$, $g_\gamma = [0.01, 0.1]$, $\rho_{\gamma x} = 0.23$, $\rho_{Vx} = -0.1$, $\beta = 0.05$, $\alpha = 5$, $t = [0, 10]$, $g_t = [0, 1]$.

The optimal policies for the four control variables – risky asset allocation $\pi(t)$, debt ratio $\tilde{\alpha}(t)$, consumption rate $c(t)$, and financial-market allocation proportion $p(t)$ – are computed using the closed-form expressions derived from the Hamilton–Jacobi–Bellman equation (57).The state variables considered in the simulations are:

- Wealth $X \in [1,10]$,
- Stochastic variance $V \in [0,1]$.

These ranges allow examination of both low- and high-wealth regimes, as well as tranquil versus turbulent market conditions.

Three-dimensional surface plots are used to illustrate how optimal controls respond jointly to changes in wealth and market volatility.

Figures 1-4 illustrate the optimal debt ratio $\tilde{\alpha}^*(X, V)$ as a function of investor wealth $X \in [1,10]$ and market variance $V \in [0,1]$, under different levels of risk aversion ε and housing price volatility σ_H.

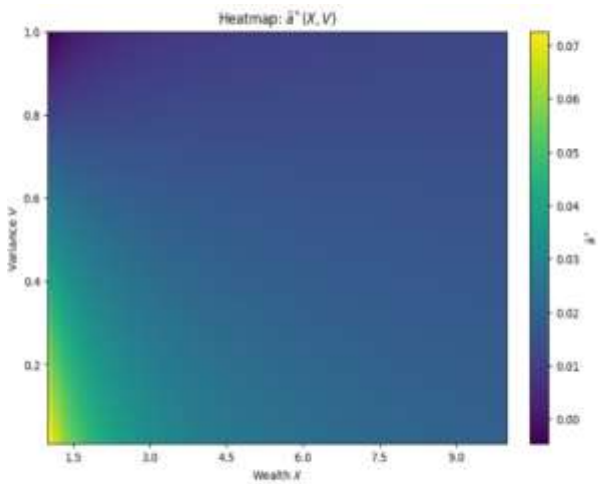

Figure 1: Optimal debt ratio of the investor for $\sigma_H = 1$ and $\varepsilon = 1.5$

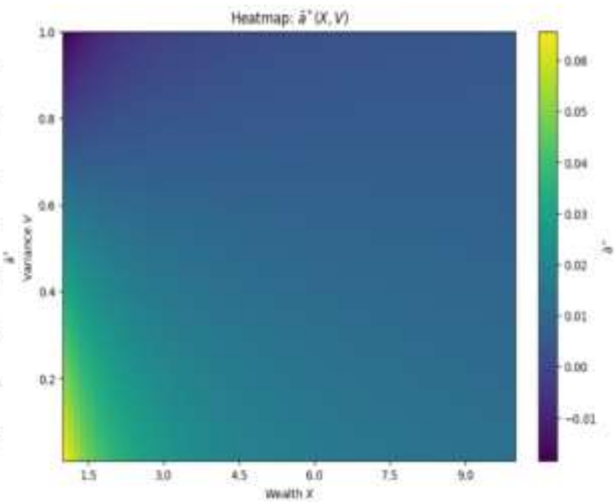

Figure 2: Optimal debt ratio of the investor for $\sigma_H = 1$ and $\varepsilon = 3$

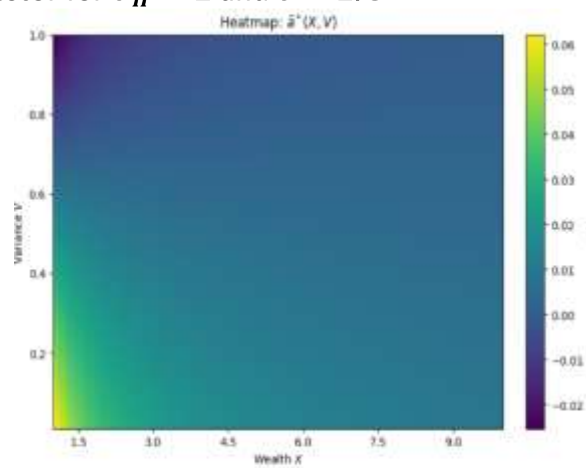

Figure 3: Optimal debt ratio of the investor for $\sigma_H = 1$ and $\varepsilon = 6$

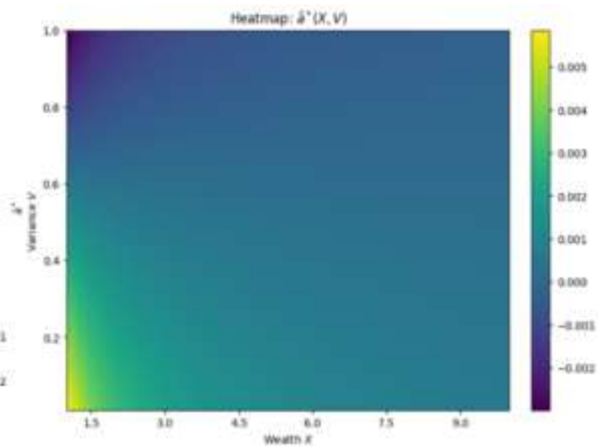

Figure 4: Optimal debt ratio of the investor for $\sigma_H = 10$ and $\varepsilon = 1.5$

We observe across all figures that the optimal debt ratio declines monotonically with wealth. At low wealth levels, borrowing serves as a leverage mechanism that allows the investor to smooth consumption and finance exposure to productive assets. As wealth increases, the marginal benefit of leverage diminishes, and the investor relies more on internal resources, leading to a lower optimal debt ratio. This behavior is consistent with classical portfolio–consumption theory under borrowing costs and reflects decreasing leverage incentives as financial constraints relax. For a fixed level of wealth, the optimal debt ratio decreases as market variance V increases. Higher volatility amplifies downside risk, making debt financing less attractive due to its asymmetric payoff structure. As volatility rises, the investor optimally reduces leverage to avoid magnifying losses during adverse market conditions. This effect is particularly pronounced at lower wealth levels, where the investor is more vulnerable to volatility shocks. Figures 1-3 reveals that higher risk aversion leads to systematically lower optimal debt ratios across the entire state space. More risk-averse investors place greater weight on downside risk and therefore choose more conservative debt policies. Figure 4 shows that higher housing volatility significantly suppresses optimal borrowing, especially in low-wealth and low-variance regions where leverage would otherwise be most attractive. This finding highlights the role of housing risk as an additional channel through which uncertainty discourages debt accumulation, reinforcing the interaction between real asset risk and financial leverage.

Figures 5-7 illustrate the investor's optimal allocation to the risky financial asset, $\pi^*(X, V)$, for wealth levels $X \in [1,10]$ and stochastic variance levels $V \in [0,1]$, under moderate stock price volatility $\sigma_S = 4$ and increasing degrees of risk aversion $\varepsilon = 1.5, 3$, and 6, respectively. Figure 8 presents the corresponding allocation when stock price volatility is substantially higher, $\sigma_S = 40$, with risk aversion fixed at ε =1.5

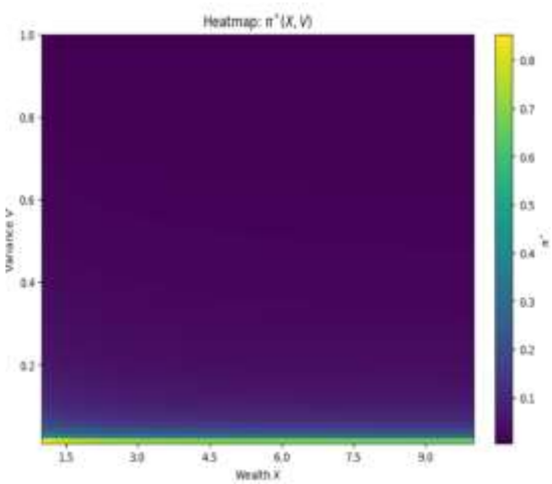

Figure 5: Optimal risky asset allocation of the investor for $\boldsymbol{\sigma_S = 4}$ and $\boldsymbol{\varepsilon = 1.5}$

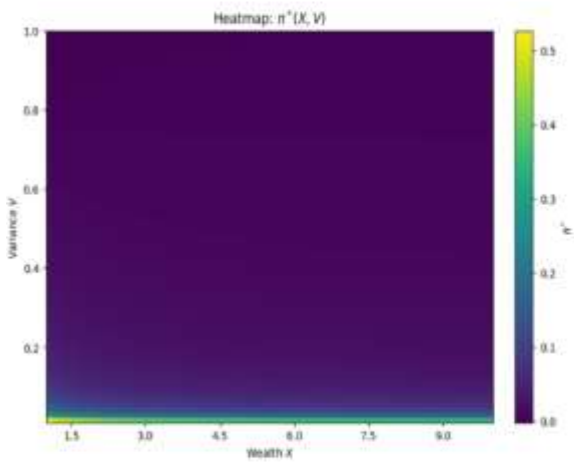

Figure 6: Optimal risky asset allocation of the investor for $\boldsymbol{\sigma_S = 4}$ and $\boldsymbol{\varepsilon = 3}$

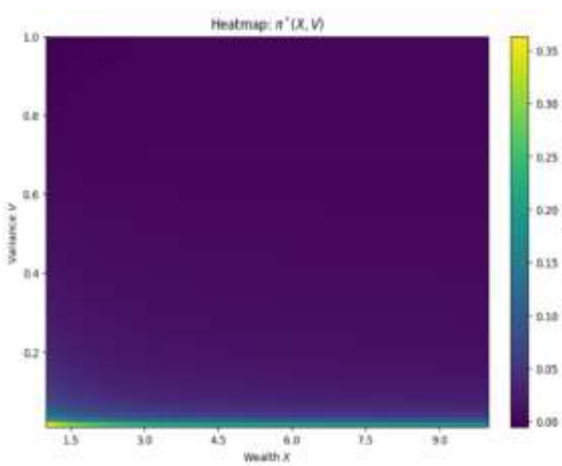

Figure 7: Optimal risky asset allocation of the investor for $\boldsymbol{\sigma_S = 4}$ and $\boldsymbol{\varepsilon = 6}$

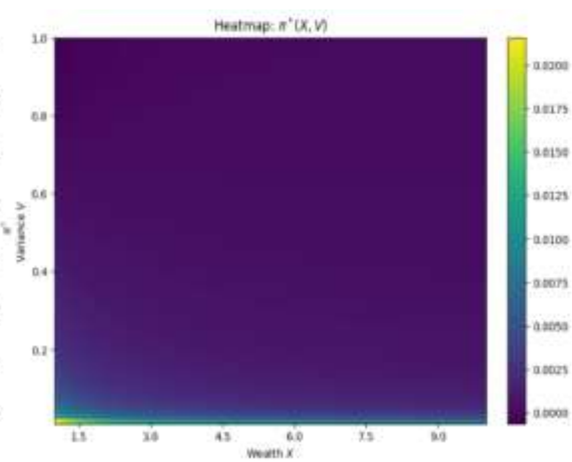

Figure 8: Optimal risky asset allocation of the investor for $\boldsymbol{\sigma_S = 40}$ and $\boldsymbol{\varepsilon = 1.5}$

We observe that across all figures, the optimal risky-asset allocation is monotonically decreasing in market variance V. As volatility rises, the investor systematically reduces exposure to the risky asset. This behavior is consistent with classical portfolio theory and stochastic-volatility models, where higher uncertainty increases the effective risk premium required to justify holding risky assets. In the present model, volatility enters both the diffusion term and the intertemporal hedging component of the optimal policy, amplifying its dampening effect on π^*. The heatmaps show that for low volatility levels $V \approx 0$, risky-asset allocation is positive and economically meaningful, whereas for higher volatility levels, the allocation rapidly declines toward zero, indicating a flight-to-safety response. For a fixed level of volatility, the risky-asset allocation exhibits weak sensitivity to wealth, particularly at moderate and high volatility levels. This is consistent with the CRRA preference structure, under which relative portfolio proportions are largely scale-invariant. Small variations observed at very low wealth levels reflect precautionary behavior driven by borrowing and consumption considerations embedded in the wealth dynamics. Figures 5, 6, and 7 reveal a clear and systematic effect of risk aversion. As ε increases from 1.5 to 6, the overall magnitude of π^* decreases sharply across all wealth and volatility regimes. Highly risk-averse investors allocate only a negligible fraction of wealth to the risky asset, even when market volatility is low. This confirms that risk aversion dominates speculative incentives in the presence of stochastic volatility and debt considerations. Figure 8 demonstrates the impact of extreme stock price volatility. When σ_S increases from 10 to 100, the optimal risky-asset allocation collapses toward zero throughout the entire state space, even for relatively low market variance V. This result highlights the strong interaction between instantaneous volatility and stochastic variance: high stock-specific risk overwhelms the expected return advantage of risky assets, rendering equity investment unattractive regardless of wealth level.

Figures 9-11 illustrate the investor's optimal consumption policy $c^*(X, V)$ as a function of wealth $X \in [1, 10]$ and stochastic market variance $V \in [0, 1]$, for a fixed wealth-risk aversion parameter $\varepsilon = 1.5$ and three values of the CRRA consumption parameter $\alpha = 1.5$, 3, and 6, respectively

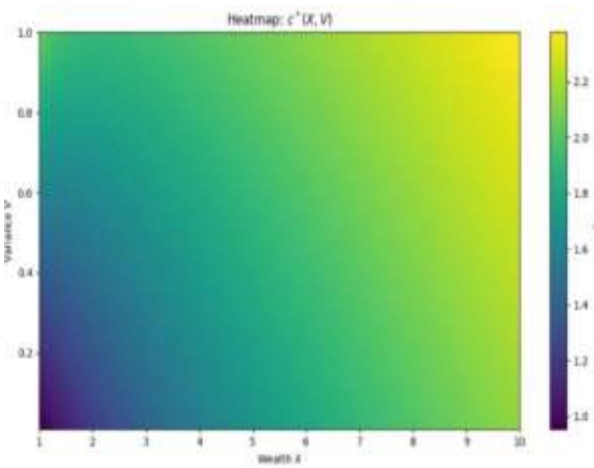

Figure 9: Optimal consumption plan of the investor $\boldsymbol{\alpha = 1.5}$

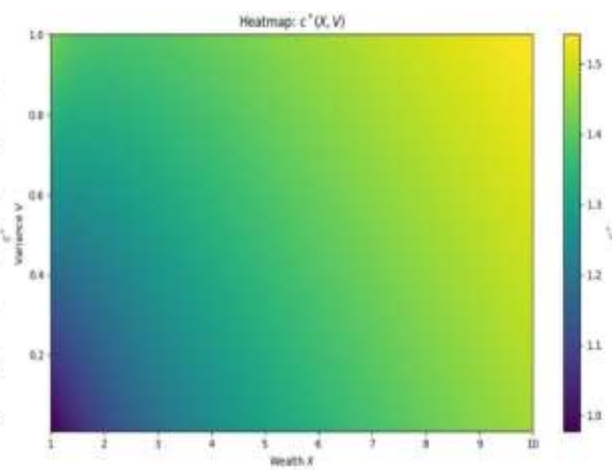

Figure 10: Optimal consumption plan of the investor for $\boldsymbol{\alpha = 3}$

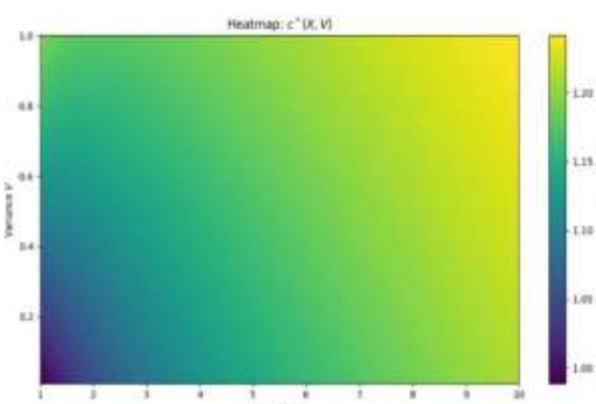

Figure 11: Optimal consumption plan of the investor for $\boldsymbol{\alpha = 6}$

Across all three figures, optimal consumption is monotonically increasing in wealth. For any fixed volatility level, higher wealth states lead to higher consumption rates. This behavior is consistent with standard CRRA preferences and confirms that the derived policy respects the fundamental economic principle that consumption scales positively with available resources. The gradient of consumption with respect to wealth becomes progressively flatter as α increases. When $\alpha = 1.5$, consumption responds strongly to changes in wealth, whereas for $\alpha = 6$, consumption increases more conservatively. This reflects stronger intertemporal smoothing incentives under higher consumption risk aversion. For any fixed wealth level, optimal consumption increases mildly with increasing market volatility. Regions of high variance V are associated with higher consumption intensity, indicating that higher volatility can induce a reallocation toward present consumption rather than precautionary saving. The investor optimally consumes more than invest when market conditions are volatile. The figures also reveal a clear risk-aversion ordering:

- At a low $\alpha = 1.5$, consumption is relatively high and more responsive to both wealth and volatility.
- At a moderate $\alpha = 3$, consumption levels decline and become smoother across states.
- At a high $\alpha = 6$, consumption is markedly conservative, with weaker sensitivity to wealth and volatility.

Higher values of α amplify the investor's preference for smoothing consumption over time, leading to systematically lower consumption rates across the state space.

CONCLUSION

This paper developed a continuous-time stochastic control framework for jointly determining optimal investment, consumption, and debt-financing decisions in an economy characterized by stochastic volatility and housing market risk. Unlike classical portfolio selection models that assume constant volatility and exclude leverage-housing interactions, the proposed model integrates a Heston-type stochastic volatility process with a housing asset and an endogenous debt ratio. Within this unified setting, the investor optimally allocates wealth among a risk-free asset, a risky financial asset, and housing, while simultaneously choosing consumption and borrowing policies under CRRA preferences. Using the dynamic programming approach, the associated Hamilton-Jacobi-Bellman equation was derived and solved explicitly. Closed-form expressions were obtained for the optimal risky asset allocation, optimal debt ratio, and optimal consumption rule. The analytical results demonstrate that stochastic volatility plays a central role in shaping both portfolio and leverage decisions. In particular, higher market volatility reduces optimal exposure to risky financial assets and lowers the optimal debt ratio, reflecting the investor's precautionary response to increased uncertainty. Housing price volatility was shown to be a key determinant of leverage, with optimal borrowing decreasing as housing risk intensifies. Consumption decisions, in contrast, are primarily driven by risk aversion, with volatility exerting only an indirect effect through its impact on wealth dynamics. The numerical analysis complemented the theoretical results by illustrating the sensitivity of optimal policies to changes in volatility, risk aversion, and asset price uncertainty. The simulations confirmed that increases in either financial or housing volatility lead to more conservative investment and borrowing behavior, while higher risk aversion shifts the investor's preference toward consumption and away from risky exposure. These findings reinforce the economic intuition of the model and highlight the importance of incorporating stochastic volatility and housing risk when evaluating optimal financial behavior. Overall, this study contributes to the literature by extending continuous-time portfolio theory to a richer and more realistic setting that captures the interaction between stochastic volatility, housing assets, and debt financing. The results provide insights into how investors optimally adjust consumption, leverage, and portfolio composition in volatile markets, particularly in economies where housing represents a significant component of household wealth.

REFERENCES

[1] Merton, R. C. (1969). “Lifetime portfolio selection under uncertainty: the continuous-time case,” The Review of Economics and Statistics, vol. 51, no. 3, pp. 247–257.

[2] Merton, R. C. (1971). “Optimum consumption and portfolio rules in a continuous-time model,” Journal of Economic Theory, vol. 3, no. 4, pp. 373–413.

[3] Fleming, W. H. and Zariphopoulou, T. (1991). “An optimal investment/ consumption model with borrowing,” Mathematics of Operations Research, vol. 16, no. 4, pp. 802–822.

[4] Vila, J. L. and Zariphopoulou, T. (1997). “Optimal consumption and portfolio choice with borrowing constraints,” Journal of Economic Theory, vol. 77, no. 2, pp. 402–431.

[5] Dumas, B. and Luciano, E. (1991). “An exact solution to a dynamic portfolio choice problem under transactions costs,” The Journal of Finance, vol. 46, no. 2, pp. 577–595.

[6] Shreve, S. E. and Soner, H. M. (1994). “Optimal investment and consumption with transaction costs,” Annals of Applied Probability, vol. 4, no. 3, pp. 609–692.

[7] Liu, H. and Loewenstein, M. (2002). “Optimal portfolio selection with transaction costs and finite horizons,” The Review of Financial Studies, vol. 15, no. 3, pp. 805–835.

[8] Dai, M., Jiang, L., Li, P., and Yi, F. (2009). “Finite horizon optimal investment and consumption with transaction costs,” SIAM Journal on Control and Optimization, vol. 48, no. 2, pp. 1134–1154.

[9] Heston, S. L. (1993). “A closed-form solution for options with stochastic volatility, with applications to bond and currency options,” The Review of Financial Studies, vol. 6, no. 2, pp. 327–343.

[10] Fleming, W. H. and Herna´ndez-Herna´ndez, D. (2003). “An optimal consumption model with stochastic volatility,” Finance and Stochastics, vol. 7, no. 2, pp. 245–262.

[11] Kraft, H. (2005). “Optimal portfolios and Heston's stochastic volatility model: an explicit solution for power utility,” Quantitative Finance, vol. 5, no. 3, pp. 303–313.

[12] Chacko, G. and Viceira, L. M. (2005). “Dynamic consumption and portfolio choice with stochastic volatility in incomplete markets,” The Review of Financial Studies, vol. 18, no. 4, pp. 1369–1402.

[13] Liu, J. (2007). “Portfolio selection in stochastic environments,” The Review of Financial Studies, vol. 20, no. 1, pp. 1–39.

[14] Bank, P., Riedel, F. (2001). “Optimal consumption choice with intertemporal substitution”. Annals of Applied Probability, vol. 11, no. 3, pp. 750–788.

[15] Jin, Z. (2014). "Optimal debt ratio and consumption strategies in financial crisis". Journal of Optimization Theory and Applications, vol. 166, no. 3 pp. 1029–1050.

[16] Liu, W. Jin, Z. (2014). "Analysis of optimal debt ratio in a Markov regime switching model". Working paper, Department of Finance, Latrobe university, Melbourne, Australia.

[17] Nkeki, C. I. (2018). "Optimal investment strategy with dividend paying and proportional transaction costs". Annals of Financial Economics, vol. 13, no. 1, Article 1850001, http://doi.org/10.1142.s201049521850001x

[18] Nkeki, C.I. (2018). "Optimal investment risks management strategies of an economy in a financial crisis". International Journal of Financial Engineering, vol. 5, no. 1, pp. 1–24, https://doi.org/10.1142/s2424786318500032.

[19] Samuelson, P. (1973). "Proof that Properly Discounted Present Values of Assets Vibrate Randomly". Bell Journal of Economic Management 4 (2), pp 369 – 374. https://doi.org/10.2307/3003046

[20] Cox, J. C. Ingersoll, J. E. Jr., and Ross, S. A. (1985). "A theory of the term structure of interest rates," Econometrica. Journal of the Econometric Society, vol. 53, no. 2, pp. 385–407.