



SOME FIXED POINT THEOREM IN QUASI METRIC SPACES WITH BINARY OPERATION

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ABSTRACT

The concept of metric space with binary operation was introduced by Adewale et al. in 2025 [1]. This is a better version of metric space. It unifies metric space and many other metric-like spaces. In the same year, Ayodele et al. introduced quasi metric space with binary operation due to the fact that distance between two points may not be the same. He also established and proved some fixed point theorems. In this paper, we establish and prove some fixed point theorems in quasi metric space with binary operation. Our work unify, extend and generalize existing work in literature.

1. INTRODUCTION

Maurice Fréchet [17] introduced the concept of a **metric space** in his doctoral thesis. A metric space is a set V with a distance function $d: V \times V \rightarrow \mathbb{R}^+$ satisfying Positivity, Symmetry and Triangle inequality [17]. In 1922, Banach contraction principle was introduced by Stefan Banach [13]. This was used as alternative method to prove the existence and uniqueness of solutions of ODE. In light of this, many authors picked interest in this area of study. Some of them generalized metric space while some either worked on contraction or iterative method used (see [1-20]). In 2025, Adewale et al. [1] introduced metric space with binary operation. This research work introduces metric space with algebraic operations like addition, multiplication and any other operations. This concept leads to combined structures like metric groups and topological groups. The concept of metric space with binary operation introduced by Adewale et al. [1] was later generalized to quasi metric space with binary operation by Ayodele et al. [10] in the same year.

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2. PRELIMINARIES

Definition 2.1 [1]

Let K be a non-empty set, ∇ , a binary operation with e as its identity element and $\rho : K \times K \rightarrow \mathbb{R}^+$. ρ is called an operational metric or metric with binary operation if the following axioms are satisfied:

$$\rho_1: \rho(m, n) \geq e;$$

$$\rho_2: \rho(m, n) = e \text{ if and only if } m = n;$$

$$\rho_3 : \rho(m, n) = \rho(n, m);$$

$$\rho_4 : \rho(m, n) \leq \rho(m, r) \nabla b(r, n) \text{ for all } m, n, r \in K.$$

K together with ρ is called an operational metric space or metric space with binary operation. It is denoted by (K, ρ, ∇) .

Remark 2.2:

- i. If the binary operation ∇ is defined by $p \nabla q = p + q$, the Definition 2.1 reduces to metric space introduced by Fretchet (1906) [17].
- ii. If the binary operation ∇ is defined by $p \nabla q = p \times q$, the Definition 2.1 reduces to multiplicative metric space introduced by **Bashirov et al** (2008) [14].

Definition 2.3 [1]

Let (K, ρ, ∇) be an operational metric space and $\{a_n\}$, a sequence in K . A sequence, $\{a_n\}$ converge to a if for $n \in \mathbb{N}, \rho(a_n, a) \rightarrow e$ as $n \rightarrow \infty$.

Definition 2.4 [1]

Let (K, ρ, ∇) be an operational metric space and $\{a_n\}$, a sequence in K . A sequence, $\{a_n\}$ in K is said to be a Cauchy sequence if for $n, m \in \mathbb{N}$ with $n > m, \rho(a_n, a_m) \rightarrow e$ as $n, m \rightarrow \infty$.

Definition 2.5 [10]

Let K be a non-empty set, ∇ , a binary operation with e as its identity element and $\rho : K \times K \rightarrow \mathbb{R}^+$. ρ is called an operational quasi metric or quasi metric with binary operation if the following axioms are satisfied:

$$\rho_1: \rho(m, n) \geq e;$$

$$\rho_2: \rho(m, n) = e \text{ if and only if } m = n;$$

$$\rho_3 : \rho(m, n) \leq \rho(m, r) \nabla b(r, n) \text{ for all } m, n, r \in K.$$

K together with ρ is called an operational quasi metric space or quasi metric space with binary operation. It is denoted by (K, ρ, ∇) .

Remark 2.6:

- i. If the binary operation ∇ is defined by $p \nabla q = p + q$, the Definition 2.3 reduces to quasi metric space introduced by Wilson (1931) [20].
- ii. If the binary operation ∇ is defined by $p \nabla q = p \times q$, the Definition 2.1 reduces to multiplicative quasi metric space.

Definition 2.7 [10]

Let (Z, b, \odot) be a quasi metric space with binary operation, \odot and $\{x_n\}$, a sequence in Z . A sequence, $\{x_n\}$ is R-converge to t if for $n \in \mathbb{N}, b(x_n, t) \rightarrow e$ as $n \rightarrow \infty$.

Definition 2.8 [10]

Let (Z, b, \odot) be a quasi metric space with binary operation, \odot and $\{x_n\}$, a sequence in Z . A sequence, $\{x_n\}$ is L-converge to t if for $n \in N$, $b(t, x_n) \rightarrow e$ as $n \rightarrow \infty$.

Definition 2.9 [10]

Let (Z, b, \odot) be a quasi metric space with binary operation, \odot and $\{x_n\}$, a sequence in Z . A sequence, $\{x_n\}$ converge to t if it is both R-convergent and L-convergent to t .

Definition 2.10 [10]

Let (Z, b, \odot) be a quasi metric space with binary operation, \odot and $\{x_n\}$, a sequence in Z . A sequence, $\{x_n\}$ in Z is said to be a Cauchy sequence if for $n, m \in N$, $b(x_n, x_m) \rightarrow e$ as $n, m \rightarrow \infty$.

3. MAIN RESULTS

In 2025, Ayodele et.al. introduced quasi metric space with binary operation. In this new space, they established and proved some fixed point theorems which did not include Ćirić fixed point theorem and Ćirić fixed point theorem is a generalisation of those contractions in the article. In this paper, we introduced more fixed point theorems in the space. This includes Ćirić like fixed point theorem.

Theorem 3.1

Let (X, d, Δ) be a complete quasi metric space with binary operation defined by $a\Delta b = a + b$. Suppose $\beta : X \rightarrow X$ is a self-map and there exists a constant $c \in [0, 0.5)$ such that for all $a, b \in X$, the following inequality holds:

$$d(\beta p, \beta q) \leq c \cdot \min\{d(p, q), d(p, \beta q), d(q, \beta p), d(p, \beta p)\}.$$

Then β has a point $s \in X$: $\beta s = s$.

Proof:

Considering $d(\beta p, \beta q) \leq c \cdot \min\{d(p, q), d(p, \beta q), d(q, \beta p), d(p, \beta p)\}$ with an arbitrary point $a_0 \in X$ and define a sequence $\{a_n\}_{n=1}^{\infty}$ by $a_n = \beta a_{n-1} = \beta^n a_0$, we obtain

$$\begin{aligned} d(a_n, a_{n+1}) &= d(\beta a_{n-1}, \beta a_n) \\ &\leq c \cdot \min\{d(a_{n-1}, a_n), d(a_{n-1}, \beta a_{n-1}), d(a_{n-1}, \beta a_n), d(a_n, \beta a_{n-1})\} \\ &= c \cdot \min\{d(a_{n-1}, a_n), d(a_{n-1}, a_{n+1})\}. \end{aligned}$$

Case I

If $d(a_{n-1}, a_n)$ is the minimum, then

$$\begin{aligned} d(a_n, a_{n+1}) &\leq c \cdot d(a_{n-1}, a_n) \\ \Rightarrow d(a_n, a_{n+1}) &\leq c^2 \cdot d(a_{n-2}, a_{n-1}) \\ \Rightarrow d(a_n, a_{n+1}) &\leq c^3 \cdot d(a_{n-3}, a_{n-2}) \\ \Rightarrow d(a_n, a_{n+1}) &\leq c^n \cdot d(a_0, a_1). \end{aligned}$$

Taking the limit of $d(a_n, a_{n+1})$ as $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} d(a_n, a_{n+1}) = \lim_{n \rightarrow \infty} c^n \cdot d(a_0, a_1) = 0.$$

So, as $n \rightarrow \infty$, $d(a_n, a_{n+1}) \rightarrow 0$.

Case II

If $d(a_{n-1}, a_{n+1})$ is the minimum, then

$$\begin{aligned} d(a_n, a_{n+1}) &\leq c \cdot d(a_{n-1}, a_{n+1}) \\ \Rightarrow d(a_n, a_{n+1}) &\leq c[d(a_{n-1}, a_n)\Delta d(a_n, a_{n+1})] \\ \Rightarrow d(a_n, a_{n+1}) &\leq c[d(a_{n-1}, a_n) + d(a_n, a_{n+1})] \\ \Rightarrow d(a_n, a_{n+1}) &\leq c \cdot d(a_{n-1}, a_n) + c \cdot d(a_n, a_{n+1}) \\ \Rightarrow d(a_n, a_{n+1}) - c \cdot d(a_n, a_{n+1}) &\leq c \cdot d(a_{n-1}, a_n) \\ \Rightarrow (1 - c)d(a_n, a_{n+1}) &\leq c \cdot d(a_{n-1}, a_n) \\ \Rightarrow d(a_n, a_{n+1}) &\leq \frac{c}{1 - c} d(a_{n-1}, a_n) \end{aligned}$$

Let $q = \frac{c}{1 - c} < 1$, then

$$\begin{aligned} d(a_n, a_{n+1}) &\leq q \cdot d(a_{n-1}, a_n) \\ \Rightarrow d(a_n, a_{n+1}) &\leq q^2 \cdot d(a_{n-2}, a_{n-1}) \\ \Rightarrow d(a_n, a_{n+1}) &\leq q^3 \cdot d(a_{n-3}, a_{n-2}) \\ \Rightarrow d(a_n, a_{n+1}) &\leq q^n \cdot d(a_0, a_1) \end{aligned} \tag{1}$$

Taking the limit of $d(a_n, a_{n+1})$ as $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} d(a_n, a_{n+1}) = \lim_{n \rightarrow \infty} q^n \cdot d(a_0, a_1) = 0.$$

So, as $n \rightarrow \infty$, $d(a_n, a_{n+1}) \rightarrow 0$.

Using ρ_4 of Definition 2.1 with $n > m$, we have

$$\begin{aligned} d(a_m, a_n) &\leq d(a_m, a_{m+1})\Delta d(a_{m+1}, a_n) \\ \Rightarrow d(a_m, a_n) &\leq d(a_m, a_{m+1}) + d(a_{m+1}, a_n) \\ \Rightarrow d(a_m, a_n) &\leq d(a_m, a_{m+1}) + d(a_{m+1}, a_{m+2})\Delta d(a_{m+2}, a_n) \\ \Rightarrow d(a_m, a_n) &\leq d(a_m, a_{m+1}) + d(a_{m+1}, a_{m+2}) + d(a_{m+2}, a_n) \end{aligned}$$

$$\Rightarrow d(a_m, a_n) \leq d(a_m, a_{m+1}) + d(a_{m+1}, a_{m+2}) + \dots + d(a_{n-1}, a_n) \quad (2)$$

With (1) and (2), we get

$$d(a_m, a_n) \leq q^m \cdot d(a_0, a_1) + q^{m+1} \cdot d(a_0, a_1) + \dots + q^{n-1} \cdot d(a_0, a_1)$$

$$\Rightarrow d(a_m, a_n) \leq (q^m + q^{m+1} + q^{m+2} + \dots + q^{n-1})d(a_0, a_1)$$

As $n \rightarrow \infty$, we have

$$d(a_m, a_n) \leq \left(\frac{q^m}{1-q} \right) d(a_0, a_1).$$

As $m \rightarrow \infty$, $d(a_m, a_n) \rightarrow 0$.

So, $\{a_n\}$ is a Cauchy sequence.

By the completeness of (X, d, Δ) , there exists $s \in X$ such that $\{a_n\}$ is R-convergent to s .

Suppose $\beta s \neq s$, then

$$d(a_n, \beta s) \leq c \cdot \min\{d(a_{n-1}, s), d(a_{n-1}, \beta a_{n-1}), d(s, \beta s), d(s, \beta a_{n-1}), d(a_{n-1}, \beta s)\}$$

Taking the limit as $n \rightarrow \infty$ and using the fact that the function is continuous in its variables, we get

$$d(s, \beta s) \leq c \cdot \min\{d(s, \beta s)\} = c \cdot d(s, \beta s)$$

Hence, $d(s, \beta s) \leq 0$, a contradiction. So, $\beta s = s$

If $\{a_n\}$ is L-convergent to s ,

$$d(\beta s, a_n) \leq c \cdot \min\{d(s, a_{n-1}), d(a_{n-1}, \beta a_{n-1}), d(s, \beta s), d(s, \beta a_{n-1}), d(a_{n-1}, \beta s)\}.$$

Taking the limit as $n \rightarrow \infty$ and using the fact that the function is continuous in its variables, we get

$$d(s, \beta s) \leq c \cdot \min\{d(s, \beta s)\} = c \cdot d(s, \beta s)$$

Hence, $d(s, \beta s) \leq 0$. This is also a contradiction. So, $\{a_n\}$ is convergent to s and $\beta s = s$.

Suppose $\beta r = r$ and $\beta s = s$ with $r \neq s$, then

$$d(\beta r, \beta s) \leq c \cdot \min\{d(r, s), d(r, \beta s), d(s, \beta r), d(r, \beta r), d(s, \beta s)\}$$

Since $\beta r = r$ and $\beta s = s$, we have $d(r, s) \leq c \cdot \max\{d(r, s), d(s, r)\}$. These are obviously two cases. If $d(r, s)$ is the minimum, $d(r, s) \leq 0$ which implies that $r = s$.

Theorem 3.2

Let (X, d, Δ) be a complete quasi metric space with binary operation defined by $a\Delta b = a + b + ab$. Suppose $\beta : X \rightarrow X$ is a self-map and there exists a constant $c \in [0, 1)$ such that for all $a, b \in X$, the following inequality holds:

$$d(\beta p, \beta q) \leq c \cdot d(p, q).$$

Then β has a unique point $s \in X$: $\beta s = s$.

Proof:

Considering $d(\beta p, \beta q) \leq c \cdot d(p, q)$ with an arbitrary point $a_0 \in X$ and define a sequence $\{a_n\}_{n=1}^{\infty}$ by $a_n = \beta a_{n-1} = \beta^n a_0$, we obtain

$$d(a_n, a_{n+1}) = d(\beta a_{n-1}, \beta a_n) \leq c \cdot d(a_{n-1}, a_n)$$

$$\Rightarrow d(a_n, a_{n+1}) \leq c^2 \cdot d(a_{n-2}, a_{n-1})$$

$$\Rightarrow d(a_n, a_{n+1}) \leq c^3 \cdot d(a_{n-3}, a_{n-2})$$

$$\Rightarrow d(a_n, a_{n+1}) \leq c^n \cdot d(a_0, a_1).$$

Taking the limit of $d(a_n, a_{n+1})$ as $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} d(a_n, a_{n+1}) = \lim_{n \rightarrow \infty} c^n \cdot d(a_0, a_1) = 0.$$

So, as $n \rightarrow \infty$, $d(a_n, a_{n+1}) \rightarrow 0$.

Using ρ_4 of Definition 2.1 with $n > m$, we have

$$d(a_m, a_n) \leq d(a_m, a_{m+1})\Delta d(a_{m+1}, a_n)$$

$$\Rightarrow d(a_m, a_n) \leq d(a_m, a_{m+1}) + d(a_{m+1}, a_n) + d(a_m, a_{m+1}) \cdot d(a_{m+1}, a_n)$$

$$\Rightarrow d(a_m, a_n) \leq d(a_m, a_{m+1}) + [1 + d(a_m, a_{m+1})]d(a_{m+1}, a_n)$$

$$\Rightarrow d(a_m, a_n) \leq c^m \cdot d(a_0, a_1) + [1 + c^m \cdot d(a_0, a_1)]d(a_{m+1}, a_n)$$

As $m \rightarrow \infty$,

$$d(a_m, a_n) \leq 0 + [1 + 0]d(a_{m+1}, a_n) = d(a_{m+1}, a_n)$$

$$\Rightarrow d(a_m, a_n) \leq d(a_{m+1}, a_n)$$

$$\Rightarrow d(a_m, a_n) \leq d(a_{m+1}, a_{m+2})\Delta d(a_{m+2}, a_n)$$

$$\Rightarrow d(a_m, a_n) \leq d(a_{m+1}, a_{m+2}) + d(a_{m+2}, a_n) + d(a_{m+1}, a_{m+2}) \cdot d(a_{m+2}, a_n)$$

$$\Rightarrow d(a_m, a_n) \leq d(a_{m+1}, a_{m+2}) + [1 + d(a_{m+1}, a_{m+2})]d(a_{m+2}, a_n)$$

$$\Rightarrow d(a_m, a_n) \leq c^{m+1} \cdot d(a_0, a_1) + [1 + c^{m+1} \cdot d(a_0, a_1)]d(a_{m+2}, a_n)$$

As $m \rightarrow \infty$,

$$d(a_m, a_n) \leq 0 + [1 + 0]d(a_{m+2}, a_n) = d(a_{m+2}, a_n)$$

$$\Rightarrow d(a_m, a_n) \leq d(a_{m+2}, a_n)$$

Doing this repeatedly, we obtain

$$d(a_m, a_n) \leq d(a_{n-1}, a_n) \leq c^{n-1} \cdot d(a_0, a_1) \leq 0.$$

So, $\{a_n\}$ is a Cauchy sequence.

By the completeness of (X, d, Δ) , there exists $s \in X$ such that $\{a_n\}$ is R-convergent to s .

Suppose $\beta s \neq s$, then

$$d(a_n, \beta s) \leq c \cdot d(a_{n-1}, s)$$

Taking the limit as $n \rightarrow \infty$ and using the fact that the function is continuous in its variables, we get

$$d(s, \beta s) \leq c. d(s, s) = 0$$

Hence, $d(s, \beta s) \leq 0$, a contradiction. So, $\beta s = s$

If $\{a_n\}$ is L-convergent to s ,

$$d(\beta s, a_n) \leq c. d(s, a_{n-1}).$$

Taking the limit as $n \rightarrow \infty$ and using the fact that the function is continuous in its variables, we get

$$d(s, \beta s) \leq c. d(s, s) = 0$$

Hence, $d(s, \beta s) \leq 0$. This is also a contradiction. So, $\{a_n\}$ is convergent to s and $\beta s = s$.

Suppose $\beta r = r$ and $\beta s = s$ with $r \neq s$, then

$$d(\beta r, \beta s) \leq c. d(r, s)$$

Since $\beta r = r$ and $\beta s = s$, we have

$$d(r, s) \leq c. d(r, s)$$

$$\Rightarrow d(r, s) \leq 0$$

$$\Rightarrow r = s.$$

Theorem 3.3

Let (X, d, Δ) be a complete quasi metric space with binary operation defined by $a\Delta b = a + b + ab$. Suppose $\beta : X \rightarrow X$ is a self-map and there exists a constant $c \in [0, 0.5)$ such that for all $a, b \in X$, the following inequality holds:

$$d(\beta p, \beta q) \leq c. [d(q, \beta q) + d(p, \beta p)].$$

Then β has a unique point $s \in X$: $\beta s = s$.

Proof:

Considering $d(\beta p, \beta q) \leq c. [d(q, \beta q) + d(p, \beta p)]$ with an arbitrary point $a_0 \in X$ and define a sequence $\{a_n\}_{n=1}^{\infty}$ by $a_n = \beta a_{n-1} = \beta^n a_0$, we obtain

$$d(a_n, a_{n+1}) = d(\beta a_{n-1}, \beta a_n) \leq c. [d(a_{n-1}, \beta a_{n-1}) + d(a_n, \beta a_n)]$$

$$\Rightarrow d(a_n, a_{n+1}) \leq c. d(a_{n-1}, \beta a_{n-1}) + c. d(a_n, \beta a_n)$$

$$\Rightarrow d(a_n, a_{n+1}) \leq c. d(a_{n-1}, a_n) + c. d(a_n, a_{n+1})$$

$$\Rightarrow d(a_n, a_{n+1}) - c. d(a_n, a_{n+1}) \leq c. d(a_{n-1}, a_n)$$

$$\Rightarrow (1 - c)d(a_n, a_{n+1}) \leq c. d(a_{n-1}, a_n)$$

$$\Rightarrow d(a_n, a_{n+1}) \leq \frac{c}{1 - c} . d(a_{n-1}, a_n)$$

Let $k = \frac{c}{1 - c} < 1$, then

$$\Rightarrow d(a_n, a_{n+1}) \leq k. d(a_{n-1}, a_n)$$

$$\Rightarrow d(a_n, a_{n+1}) \leq k^2. d(a_{n-2}, a_{n-1})$$

$$\Rightarrow d(a_n, a_{n+1}) \leq k^3 \cdot d(a_{n-3}, a_{n-2})$$

$$\Rightarrow d(a_n, a_{n+1}) \leq k^n \cdot d(a_0, a_1).$$

Taking the limit of $d(a_n, a_{n+1})$ as $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} d(a_n, a_{n+1}) = \lim_{n \rightarrow \infty} k^n \cdot d(a_0, a_1) = 0.$$

So, as $n \rightarrow \infty$, $d(a_n, a_{n+1}) \rightarrow 0$.

Using ρ_4 of Definition 2.1 with $n > m$, we have

$$d(a_m, a_n) \leq d(a_m, a_{m+1})\Delta d(a_{m+1}, a_n)$$

$$\Rightarrow d(a_m, a_n) \leq d(a_m, a_{m+1}) + d(a_{m+1}, a_n) + d(a_m, a_{m+1}) \cdot d(a_{m+1}, a_n)$$

$$\Rightarrow d(a_m, a_n) \leq d(a_m, a_{m+1}) + [1 + d(a_m, a_{m+1})]d(a_{m+1}, a_n)$$

$$\Rightarrow d(a_m, a_n) \leq k^m \cdot d(a_0, a_1) + [1 + k^m \cdot d(a_0, a_1)]d(a_{m+1}, a_n)$$

As $m \rightarrow \infty$,

$$d(a_m, a_n) \leq 0 + [1 + 0]d(a_{m+1}, a_n) = d(a_{m+1}, a_n)$$

$$\Rightarrow d(a_m, a_n) \leq d(a_{m+1}, a_n)$$

$$\Rightarrow d(a_m, a_n) \leq d(a_{m+1}, a_{m+2})\Delta d(a_{m+2}, a_n)$$

$$\Rightarrow d(a_m, a_n) \leq d(a_{m+1}, a_{m+2}) + d(a_{m+2}, a_n) + d(a_{m+1}, a_{m+2}) \cdot d(a_{m+2}, a_n)$$

$$\Rightarrow d(a_m, a_n) \leq d(a_{m+1}, a_{m+2}) + [1 + d(a_{m+1}, a_{m+2})]d(a_{m+2}, a_n)$$

$$\Rightarrow d(a_m, a_n) \leq k^{m+1} \cdot d(a_0, a_1) + [1 + k^{m+1} \cdot d(a_0, a_1)]d(a_{m+2}, a_n)$$

As $m \rightarrow \infty$,

$$d(a_m, a_n) \leq 0 + [1 + 0]d(a_{m+2}, a_n) = d(a_{m+2}, a_n)$$

$$\Rightarrow d(a_m, a_n) \leq d(a_{m+2}, a_n)$$

Doing this repeatedly, we obtain

$$d(a_m, a_n) \leq d(a_{n-1}, a_n) \leq k^{n-1} \cdot d(a_0, a_1) \leq 0.$$

So, $\{a_n\}$ is a Cauchy sequence.

By the completeness of (X, d, Δ) , there exists $s \in X$ such that $\{a_n\}$ is R-convergent to s .

Suppose $\beta s \neq s$, then

$$d(a_n, \beta s) \leq c[d(a_{n-1}, a_n) + d(s, \beta s)].$$

Taking the limit as $n \rightarrow \infty$ and using the fact that the function is continuous in its variables, we get

$$d(s, \beta s) \leq c[d(s, s) + d(s, \beta s)] \Rightarrow d(s, \beta s) \leq c \cdot d(s, \beta s) \Rightarrow (1 - c)d(s, \beta s) \leq 0.$$

Hence, $d(s, \beta s) \leq 0$, a contradiction. So, $\beta s = s$

If $\{a_n\}$ is L-convergent to s ,

$$d(\beta s, a_n) \leq c \cdot [d(s, \beta s) + d(a_{n-1}, a_n)]$$

Taking the limit as $n \rightarrow \infty$ and using the fact that the function is continuous in its variables, we get

$$d(s, \beta s) \leq c \cdot d(s, \beta s)$$

Suppose $\beta r = r$ and $\beta s = s$ with $r \neq s$, then

$$d(\beta r, \beta s) \leq c \cdot [d(r, \beta r) + d(s, \beta s)]$$

Since $\beta r = r$ and $\beta s = s$, we have

$$d(r, s) \leq c \cdot 0$$

$$\Rightarrow d(r, s) \leq 0$$

$$\Rightarrow r = s.$$

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