Transactions of
The Nigerian Association of
Mathematical Physics
Journal homepage: https://nampjournals.org.ng

# MATHEMATICAL MODELING AND EXACT SOLUTION OF MAGNETOHYDRODYNAMIC PRESSURE-DRIVEN FLOW AND HEAT TRANSFER BETWEEN POROUS PLATES WITH VISCOUS DISSIPATION

**Babatunde Aina[1], Ahmed Dauda Abubakar[2] and Badamasi Umar Ishaq[3]**
*[1,2,3] Department of Mathematics, Federal University, Gashua, Nigeria*
ainavicdydx@gmail.com (08063178775)

ARTICLE INFO

*Article history*:
Received xxxxx
Revised xxxxx
Accepted xxxxx
Available online xxxxx

*Keywords:*
Magnetohydrodynamic,
Pressure-Driven,
Suction/Injection,
Transverse Magnetic Field
Viscous Dissipation,

**ABSTRACT**

This study concentrates on the analytical study of the effects of magnetohydrodynamic (MHD) Pressure - driven flow and heat transfer between porous plates, taking into account the impact of viscous dissipation. The energy equation incorporates viscous dissipation effects, representing internal heat generation due to fluid friction. Because of viscous dissipation, the momentum and energy equations form a coupled system of ordinary differential equations. A parametric analysis of key physical parameters is carried out and representative numerical results for the velocity and temperature fields, as well as heat transfer rate and skin friction are presented graphically to highlight notable effects of suction and injection, Hartmann number, and viscous dissipation parameter. During the investigation, it was found that increasing the value of Hartmann number have a tendency to slow down the movement of the fluid in the porous plate. Suction and injection parameters significantly enhanced both the fluid velocity and temperature. In addition, the results show that increasing viscous dissipation significantly enhances the fluid temperature and alters the heat transfer characteristics.

## 1. Introduction

The study of Magnetohydrodynamic (MHD) transport phenomena including induction effects has gained increasing applications across various fields such as geophysics, chemical technologies, environments science, and engineering etc. When a magnetic field is applied to a fluid flow it induces an electromotive force and alters the velocity distribution through the formation of Hartmann boundary layers on surfaces perpendicular to the field.

*Corresponding author: Babatunde Aina, (08063178775)
*E-mail address:* ainavicdydx@gmail.com
https://doi.org/10.60787/tnamp.v25.713

Compared to a magnetic field applied along the direction of flow, a transvers magnetic field acts directly on the velocity of the fluid and may be more active for flow control. However, it also increases drag forces and requires greater energy input Khan [1].

Many industrial types of equipment, such as MHD generators, pumps, bearings and boundary layer control are affected by the interaction between the electrical conducting fluid and a magnetic field. The behaviour of the flow strongly depends on the orientation and intensity of the applied magnetics. The exerted magnetic field manipulates the suspended particles and rearranges their concentration in the fluid which strongly changes heat transfer characteristics of the flow.

Many studies have been conducted in the field of MHD flow. For instance, Sheikholeslami *et al.* [2] have investigated effect of the magnetic field on nanofluid flow and heat transfer in a semi-annulus enclosure via control volume based finite element method. Sheikholeslami and Gorji-and Bandpy[3] have discussed the numerical solution for free convection of ferro-fluid in a cavity heated from below in the presence of external magnetic field. The magnetohydrodynamics (MHD) boundary layer slip flow over a vertical stretching sheet in nanofluid with non-uniform heat generation/absorption in the presence of a uniform transverse magnetic field have been made by Das *et al.*[4]. The unsteady hydromagnetic flow of a viscous, incompressible, electrically conducting nanofluid past a flat plate in the presence of a uniform transverse magnetic field when the plate started impulsively from rest moves with uniform velocity in its own plane have been analyzed by Das *et al.* [5]. Jha *et al.* [6] discussed the fully developed steady natural convection flow of conducting fluid in a vertical parallel plate microchannel in the presence of transverse magnetic field. The influences of externally applied transverse magnetic field as well as suction/injection on steady natural convection flow of conducting fluid in a vertical micro-channel was carried out by Jha *et al.* [7]. They reported in their work that, the effect of suction/injection parameter on the microchannel velocity slip and temperature jump become significant with the decrease of the wall-ambient temperature difference ratio while in their work, they neglected induced magnetic. Jha and Aina [8] have obtained the exact solutions for the MHD natural convection flow in a vertical micro-porous. In other related works, Jha and Aina [9] have discussed the MHD natural convection flow in a vertical micro-porous-annulus (MPA) in the presence of radial magnetic field. The MHD natural convection flow in vertical micro-concentric-annuli (MCA) in the presence of radial magnetic field has been discussed by Jha and Aina [10]. In another related work, Jha and Aina [11] have investigated the interplay of electrically conducting and non-conducting walls on Magnetohydrodynamic mixed convection in a vertical permeable micro-channel in existence of induced magnetic field. Recently, Jayanthi & Niranja [12] studied the effects of joule heating, viscous dissipation and activation energy on nanofluid flow induced by MHD on a vertical surface which shows the effect on temperature profiles and heat transfer characteristics (local Nusselt number). Zainodin *et al.* [13] have investigated the MHD mixed convection flow of hybrid ferrofluid near a stagnation point with viscous dissipation and joule heating. Ullah *et al*. [14] demonstrates that joule heating significantly increases temperature fields in unsteady MHD micro polar flows, with viscous dissipation amplifying entropy generation. Also, Govidan [15] discussed the porosity dramatically modifies MHD boundary layer dynamics when combined with viscous and joule dissipation, enhance porosity reduce drag and alters temperature gradients. Al-Guedri *et al.* [16] have investigated the Hybrid nanofluids under magnetic fields and show that notable changes in heat transfer rates due to joule and viscous dissipation particularly relevant for thermal management systems.

On the other hand, the importance of suction/injection in boundary layer control is well established in aerodynamics and space science. (Sing) [17] and Ishak et al. [18] showed that suction/injection of a fluid through the bounding surface as for example in mass transfer cooling, can significantly change the flow field and, as a consequence, affect the heat transfer rate from the plate. Also, Al-Sanera [19] pointed out that in general, suction tends to increase the skin friction and heat transfer coefficient whereas injection acts in the opposite manner. Suction/injection plays an important role in the control of flow past an infinite plate or within parallel plates, hence its importance in practical problems involving film cooling, control of boundary layers in industrial, geophysical, biomedical, engineering and environmental applications. In another article, Shojaefard et al.[20] investigated flow control on a subsonic air foil by suction/injection and concluded that suction significantly increases the lift coefficient while injection decreases the surface skin friction, which transitively resulted in a considerable reduction in energy consumed during flights of subsonic aircrafts. To date, extensive research has been carried out on MHD natural and mixed convective heat and mass transfer in vertical channels and micro-channels. Nevertheless, to the best of the authors' knowledge no study has yet addressed MHD on pressure driven flow and heat transfer between porous plates accounting for viscous dissipation.

The main goal of the present work is to investigate the effects of magnetohydrodynamic Pressure-driven flow and heat transfer between porous plates, taking into account the impact of viscous dissipation. Accordingly, the present study provides an exact solution for MHD pressure -driven flow and heat transfer between porous plates in the presence of viscous dissipation. Moreover, exact solutions are highly important for several reasons. They serve as bench marks for assessing the accuracy of viscous provide approximate methods including numerical and empirical approaches. It is well established that exact solutions possess intrinsic theoretical significance and many have played pivotal roles in the early development of fluid mechanics and heat transfer. Beyond their theoretical significance, exact solutions can be used to assess the accuracy, convergence and efficiency of various numerical computation methods and improving differencing schemes, grid generation techniques and related computational procedures. Exact solutions are therefore highly valuable, even in the context of rapidly advancing computational fluid dynamics and heat transfer problems.

## MATHEMATICAL ANALYSIS

Effect of Magnetohydrodynamics pressure-driven flow and heat transfer between porous plates in the presence of viscous dissipation is considered. The geometry of the problem is shown in Figure 1, where axis is taken vertically upward along the plates and the axis normal to it. The distance between the plates is . Fluid is being injected into the flow region through the hot porous plate and in order to conserve the mass of the fluid in the porous plate, fluid is being sucked out of the porous plate at the same rate through the cold porous plate.

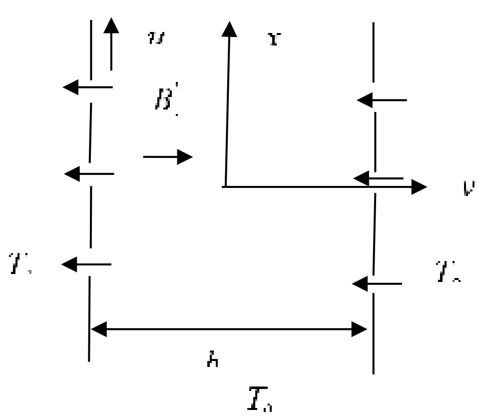


Figure 1: Flow configuration and coordinate system

The flow is assumed to be steady and fully developed, i. e., the transverse velocity is zero. A magnetic field of uniform strength $B_0^{'}$ is assumed to be applied in the direction perpendicular to the direction of flow. Both plates $y'=0$ and $y'=b$ are taken to be electrically non-conducting. As stated earlier, our emphasis in this paper is to investigate the combined effects of the magnetic field and suction/injection on pressure driven flow and heat transfer between porous plate with viscous dissipation **under constant temperature**. $x'$ - axis, then $\vec{v}=\left[u', V_0^{'}, 0\right]$ is the velocity vector and $\vec{B}=\left[0, B_0^{'}, 0\right]$ is the magnetic field vector of the considered problem.

Using the following non-dimensional quantities

$$y=\frac{y'}{b},\ U=\frac{\nu u'}{g\beta b^2\left(T_1-T_0\right)},\ \theta=\frac{T'-T_0}{T_1-T_0}, \tag{1}$$

$$\Pr=\frac{\nu}{\alpha},\ S=\frac{V_0^{'}b}{\nu},\ M^2=\frac{\sigma B_0^2 b^2}{\rho\nu}$$

the governing equations in non-dimensional form have taken the form

$$\frac{d^2U}{dy^2}+S\frac{dU}{dy}-M^2U-P=0 \tag{2}$$

$$\frac{d^2\theta}{dy^2}+S\Pr\frac{d\theta}{dy}+Ec\Pr\left[\left(\frac{dU}{dy}\right)^2\right]=0 \tag{3}$$

with the boundary conditions in non-dimensional form as

$$U(y)=0 \qquad \theta(y)=0 \qquad \text{at} \quad y=-1 \tag{4}$$

$$U(y)=0 \qquad \theta(y)=1 \qquad \text{at} \quad y=1 \tag{5}$$

The solutions for equations (2) - (3) with the boundary conditions (4) and (5) are;

$$U(y)=\exp\left(-\frac{S}{2}y\right)\left[C_1\cosh(\delta y)+C_2\sinh(\delta y)\right]+\frac{P}{M^2} \tag{6}$$

$$\theta(y)=A_{11}\exp(A_6 y)+A_{12}\exp(A_7 y)+A_{13}\exp(-sy)+A_{14}C_3+C_4\exp(-spry) \tag{7}$$

where:

$$C_1 = -\frac{P}{M^2}\frac{\cosh\left(\frac{S}{2}\right)}{\cosh(\delta)},\; C_2 = -\frac{P}{M^2}\frac{\sinh\left(\frac{S}{2}\right)}{\sinh(\delta)},\; C_3 = \frac{A_{16}A_{17} - A_{15}A_{18} - A_{16}}{A_{14}A_{18} - A_{14}A_{16}},\; C_3 = \frac{A_{17} - A_{15} - 1}{A_{16} - A_{18}}$$

Using expression (6), the skin – friction on both porous plates in dimensionless form are given by:

$$\tau_{-1} = \left.\frac{dU}{dy}\right|_{y=-1} \tag{8}$$

$$= \exp\left(\frac{S}{2}\right)\left[A_2 \cosh(\delta) - A_1 \sinh(\delta)\right] \tag{9}$$

$$\tau_1 = \left.\frac{dU}{dy}\right|_{y=1} \tag{10}$$

$$= \exp\left(-\frac{S}{2}\right)\left[A_1 \sinh(\delta) + A_2 \cosh(\delta)\right] \tag{11}$$

Using expression (7), the rate of heat transfer on both porous plates in dimensionless form is given by:

$$Nu_{-1} = \left.\frac{d\theta}{dy}\right|_{y=-1} \tag{12}$$

$$= A_{11}A_6 \exp(-A_6) + A_{12}A_7 \exp(-A_7) - A_{13}S\exp(S) - S\Pr C_4 \exp(S\Pr) \tag{13}$$

$$Nu_1 = \left.\frac{d\theta}{dy}\right|_{y=1} \tag{14}$$

$$= A_{11}A_6 \exp(A_6) + A_{12}A_7 \exp(A_7) - A_{13}S\exp(-S) - S\Pr C_4 \exp(-S\Pr) \tag{15}$$

where $A_1, A_2, ..., A_{18}$ are constants given in Appendix.

## RESULTS AND DISCUSSION

In this study, the basic parameters that govern the flow in the plates are: the suction/injection parameter $(S)$, Hartmann number $(M)$, Prandtl number $(\Pr)$, and viscous dissipation parameter $(N)$ while the numerical values of the velocity, temperature, skin friction and heat transfer rate are presented in Figs. 2– 11.

Figure 2 and 3 show the velocity and temperature profile for the different influence of suction/injection parameter $(S)$. It is clear from the graph that with suction on the cold wall of the porous channel with simultaneous injection on the hot wall of the porous channel $(S > 0)$ increases, the fluid velocity is enhanced. This is as a result of heated fluid particles being introduced into the micro-porous plates by injection through the hot plate while cold fluid particles are removed from the porous plate by suction through the cold plate. Thus, strengthening the convection current within the porous plate leading to enhancement in the fluid velocity.

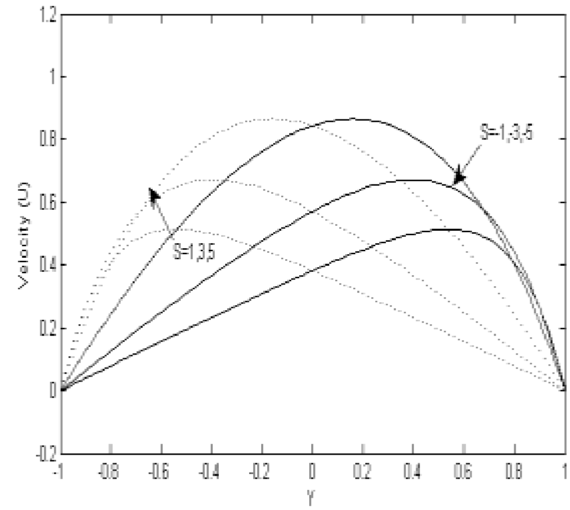


Figure 2: Variation of Velocity with $S(M = 2.0, \mathrm{Pr} = 0.71, Ec = 0.5, P = 2.0)$

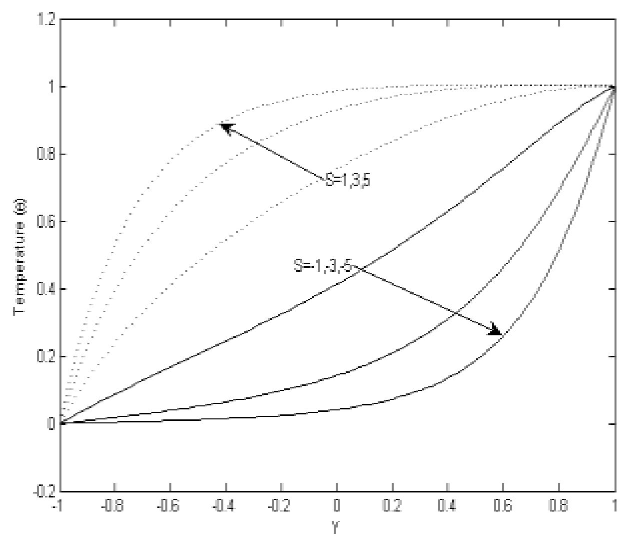


Figure 3: Variation of Temperature with $S(M = 2.0, \mathrm{Pr} = 0.71, Ec = 0.5, P = 2.0)$

Figure 4 and 5 also show the variation of velocity and temperature for various values of Hartmann number $(M)$. The graphs clearly show that increasing the value of Hartmann number have a tendency to slow down the movement of the fluid in the porous plate. Further, the shape of velocity profile has found from parabolic type to flattened type with increase in the value of Hartmann number. This is due to the fact that, the retarding effect of the magnetic force is strengthened with the increase in Hartmann number. This increase in Hartmann number, decreases the thickness of the momentum boundary layer and induces a Lorentz force which opposes the fluid

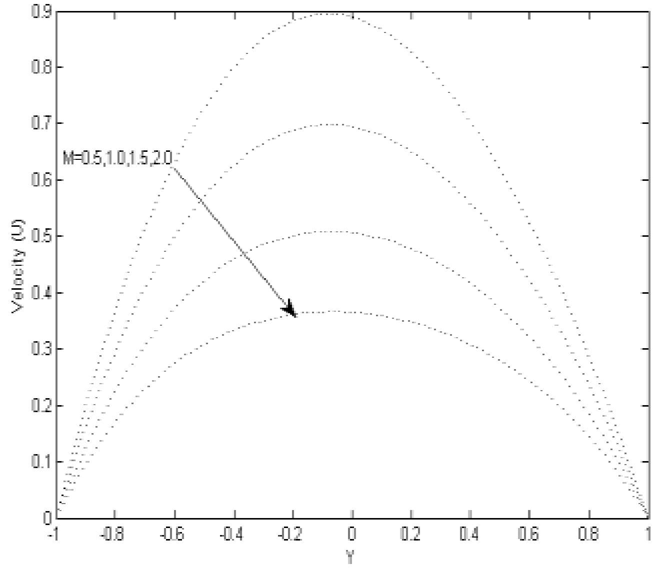


Figure 4: Variation of velocity with $M\left(S = 0.5, \mathrm{Pr} = 0.71, Ec = 0.5, P = 2.0\right)$

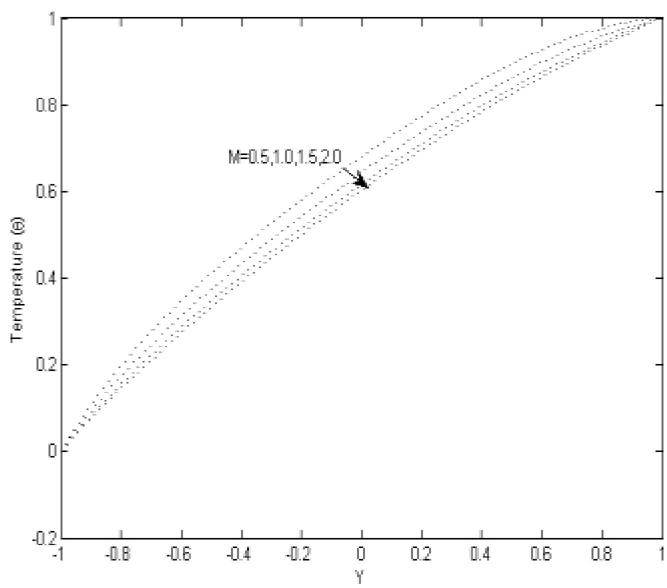


Figure 5: Variation of Temperature with $M\left(S = 0.5, \mathrm{Pr} = 0.71, Ec = 0.5, P = 2.0\right)$

Figure 6 illustrates the influence of Prandtl number $(\mathrm{Pr})$, on the temperature distribution. It is observed for Air, Water and Engine Oil, that increasing in Prandtl number leads to enhance in the fluid temperature.

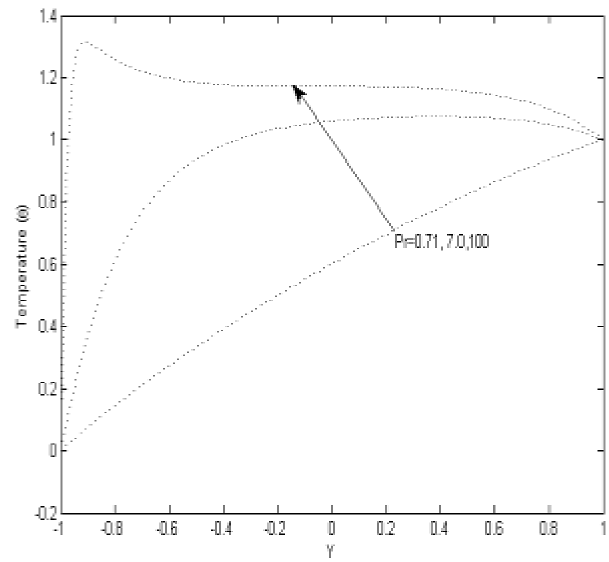


Figure 6: Variation of Temperature with $\Pr(S = 0.5, M = 0.5, Ec = 0.5, P = 2.0)$

Figure 7 display the effect of viscous dissipation parameter on the fluid temperature. It is observed that, the fluid temperature is enhanced with the increase in viscous dissipation parameter.

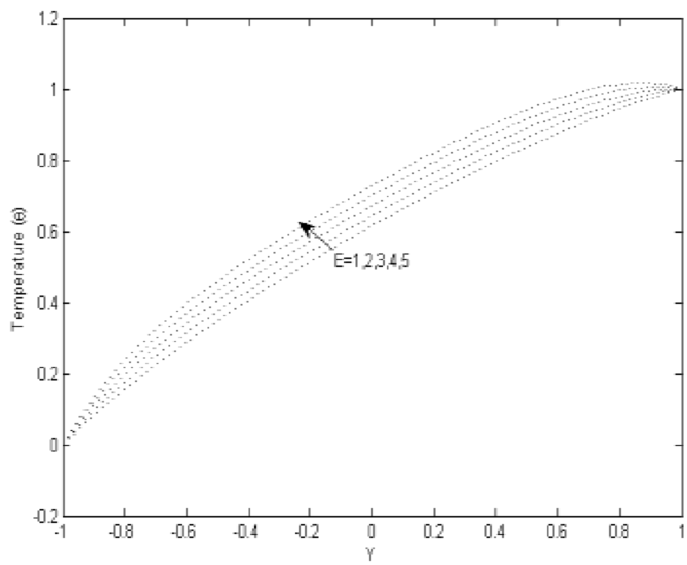


Figure 7: Variation of Temperature with $Ec(S = 0.5, M = 0.5, \Pr = 0.71, P = 2.0)$

Figures 8 and 9 illustrate the effect of suction/injection parameter, and Hartmann number on the rate of heat transfer. It is observed that, the Nusselt number reduced with increase in both suction/injection parameter, and Hartmann number at the channel wall $(y = -1,1)$

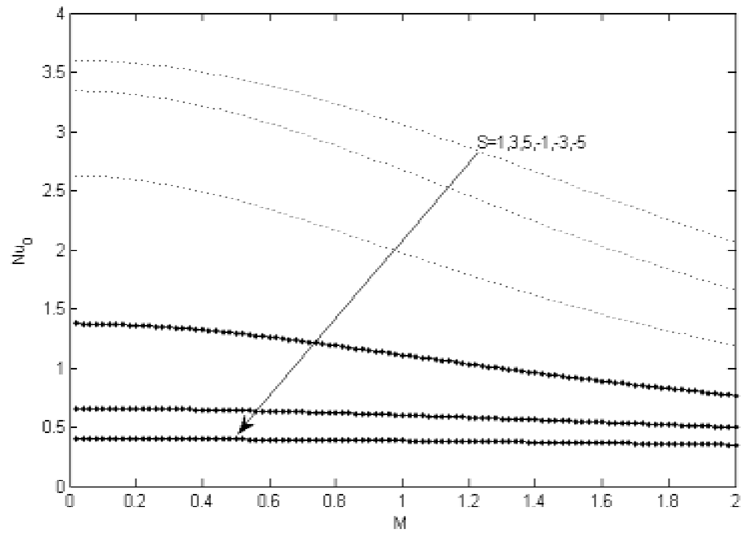


Figure 8: Effects of $S$ and $M$ on the Skin friction at $y = -1$

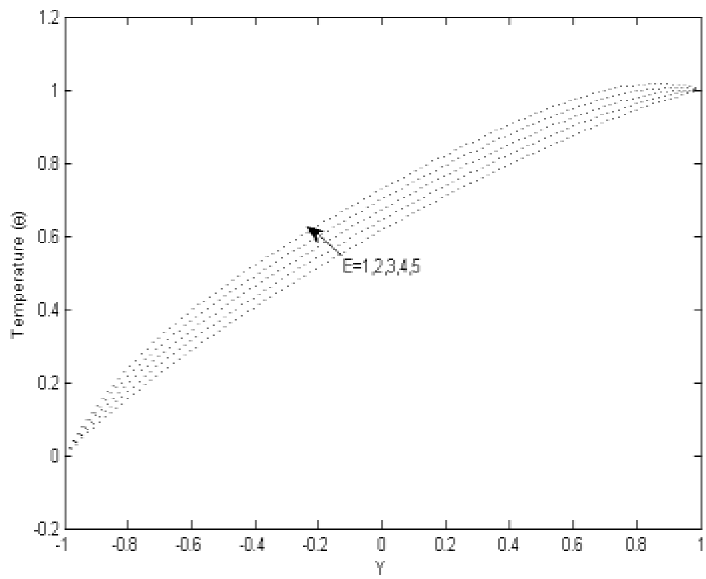


Figure 9: Effects of $S$ and $M$ on the Skin friction at $y = 1$

Figures 10 and 11 reveals the combined influences of Hartmann number and suction/injection parameter on skin friction at the channel walls $y = -1$ and $y = 1$, respectively. It is observed that, increase in Hartmann number and suction/injection parameter $(S > 0)$ causes enhancement in skin friction at the $\tau_0$ while the reverse is the case at the $\tau_1$.

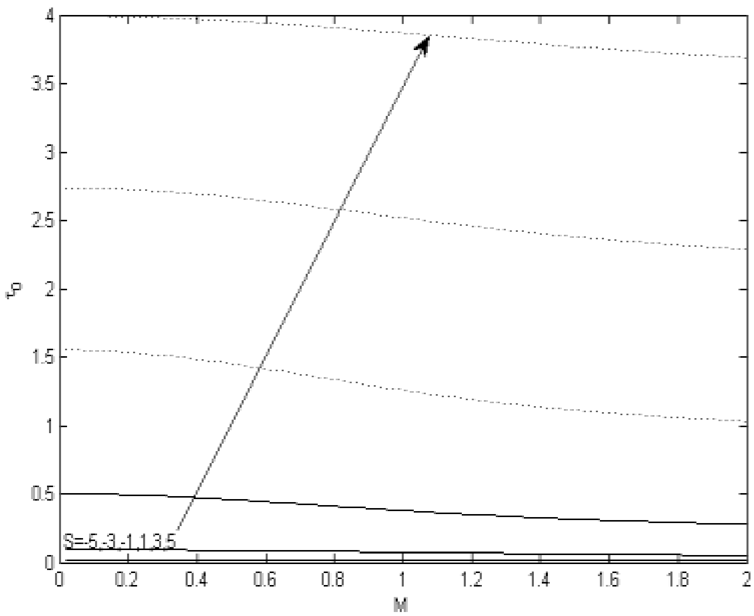


Figure 10: Effects of $S$ and $M$ on the rate of heat transfer at $y=-1$

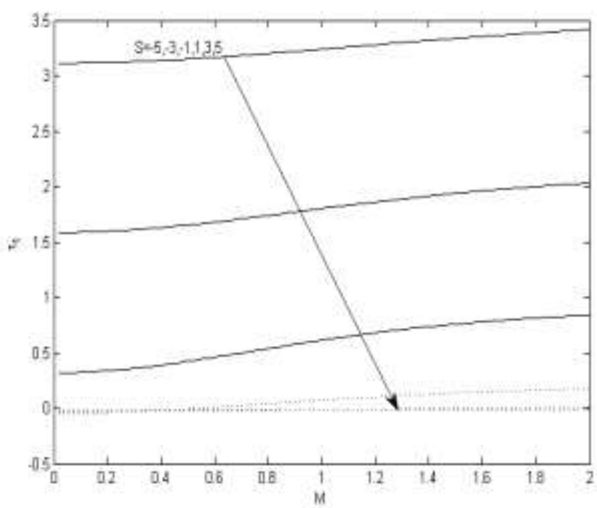


Figure 11: Effects of $S$ and $M$ on the rate of heat transfer at $y=1$

## Conclusions

In the present study, the impact of magnetohydrodynamic (MHD) on pressure driven flow and heat transfer between porous plates including the effects of viscous dissipation is investigated. The velocities, temperature, rate of heat transfer and skin friction are express as functions of the suction/injection parameter $(S)$, Hartmann number $(M)$, and magnetic Prandtl number $(\Pr)$. Base on the result obtained from the current study indicated that the following conclusions are obtained.

1. As the Hartman number increases, the magnitude of maximum value of the velocity has a decreasing tendency.
2. The influence of suction and injection parameters increased both the fluid velocity and temperature.
3. The skin friction on both channel plats reduce with an increase in suction velocity, while it also increases on both channel walls with a higher Hartmann number.
4. The fluid temperature is increase as the viscous dissipation parameter rises.
5. The effect of magnetic field on the skin friction can be useful in mechanical engineering for modelling a system. It can be easily obtained a suitable value of Hartmann number for which the value of the skin friction will be optimum.

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**Appendix**

$$\delta=\sqrt{\frac{S^2}{4}+M^2},\ A_1=C_1\delta-\frac{S}{2}C_2,\ A_2=C_2\delta-\frac{S}{2}C_1,\ A_3=-E\Pr\left(\frac{A_1^2}{2}+\frac{A_2^2}{2}\right),$$

$$A_4=-E\Pr\left(\frac{A_2^2}{2}+\frac{A_1^2}{2}\right),\ A_5=-E\Pr A_1A_2,\ A_6=2\delta-S,\ A_7=-(2\delta+S),$$

$$A_8=\frac{A_3}{2A_6}+\frac{A_5}{2A_6},\ A_9=\frac{A_3}{2A_7}-\frac{A_5}{2A_7},\ A_{10}=-\frac{A_4}{S},\ A_{11}=\frac{A_8}{A_6+S\Pr},\ A_{12}=\frac{A_9}{A_7+S\Pr},$$

$$A_{13}=\frac{A_{10}}{S\Pr-S},\ A_{14}=\frac{1}{S\Pr},\ A_{15}=A_{11}\exp(-A_6)+A_{12}\exp(-A_7)+A_{13}\exp(S),\ A_{16}=\exp(S\Pr),$$

$$A_{17}=A_{11}\exp(A_6)+A_{12}\exp(A_7)+A_{13}\exp(-S),\ A_{18}=\exp(-S\Pr).$$