Journal of TNAMP 25 (2026) 77-88

Transactions of
The Nigerian Association of
Mathematical Physics
Journal homepage: https://nampjournals.org.ng

Carbon Monoxide under the Coulomb-Hulthén-Pöschl-Teller Potential: A Quantum Stability Analysis

[1*]Ebomwonyi, O. and [2]Adebisi, A.

[1]*Department of Physics, University of Benin, Benin City, Nigeria*

[2]*Department of Physics, Edo University, Iyamho, Nigeria*

*Corresponding author's email address: osarodion.ebomwonyi@uniben.edu; +2348058865721.

ARTICLE INFO

Article history:
Received xxxxx
Revised xxxxx
Accepted xxxxx
Available online xxxxx

Keywords:
Carbon monoxide,
Coulomb-Hulthén-Pöschl-Teller potential,
Stability,
Formula Method,
energy eigenvalues.

ABSTRACT

The quantum mechanical stability of carbon monoxide molecule (CO) with the Coulomb-Hulthén-Pöschl-Teller (CHPT) potential was investigated using the Formula Method. The computed energy spectrum of the investigated quantum states lies below the dissociation threshold, suggesting persistent molecular stability over the screening interval considered. The absence of level crossing and the smooth changes of the energy eigenvalues show that the CHPT potential gives a physically consistent description of the behavior of the bound state of CO. A stability phase-diagram and a percentage stability-loss analysis were also used to quantify the stability characteristic of CO to illustrate the steady reduction in molecular binding with increase in screening. The CHPT potential results of this study compared with those reported for diatomic molecules show that it gives a reliable framework for describing the quantum stability of CO, and also offering a different approach for the investigation of bound state properties of diatomic molecules.

1. Introduction

Diatomic molecular stability of molecular systems is a fundamental problem in quantum mechanical studies because it ascertains the persistence of chemical bonds, discreet energy spectra existence, and the response of molecules to external perturbations. When the constituent atoms of a molecule remain bound within a potential well, the molecule is considered to be stable; and this gives rise to bound state energy levels that are discreet. The way these levels are distributed and characterized, provided useful information regarding molecular spectrum, vibrational dynamics and dissociation mechanisms [1-3]. This has led to the considerable interest in the investigation of molecular stability via quantum mechanical techniques in atomic, chemical and molecular physics.

*Corresponding author: Ebomwonyi, O
E-mail address: osarodion.ebomwonyi@uniben.edu, +2348058865721
DOI: https://doi.org/10.60787/tnamp.v25.720

Carbon monoxide molecule (CO) occupies a unique position among different diatomic molecules; this is because of its exceptional chemical stability, bond strength, and wide uses in astrophysics, plasma physics, combustion chemistry and environmental studies. CO is an abundant heteronuclear diatomic molecule in interstellar space which act as a useful tracer in astronomical observations [4-6]. Also, CO exhibits a strong vibrational-rotational spectrum, making it a good candidate in the testing of quantum mechanical models and molecular interaction potentials. Therefore, the exact determination of the spectrum of its bound state energy, as well as the stability characteristics remains important in theoretical and experimental research.

The description of molecular systems theoretically, depends heavily on the choice of interaction potential. Several interacting potential models have been proposed over the years, to model the interactions between atoms in diatomic molecules. There interaction potentials include the Morse potential, Kratzer potential, Hulthén potential, Manning-Rosen potential, Deng-Fan potential and the Pöschl-Teller potential [7-12]. These interaction potential models have been used to study spectra of molecules, vibrational frequencies, thermodynamic functions, and dissociation energies. However, since molecular interactions involve both short and long-range forces, these individual interaction potentials sometimes exhibit shortcomings in accurately reproducing all characteristics of molecular behavior. This limitation has led to the development of hybrid interaction potential models that combines the capacity of multiple potentials within a single framework. These hybrid interaction potential models have proved to be powerful tools in the investigation of atomic and molecular systems because they give better flexibility in the description of complex interactions. Studies using hybrid interaction potential models have shown that combination of conventional interaction potential models can easily model greater proportion of physical phenomena, such as vibrational spectra, thermodynamic properties, and confinements [13-15]. These developments created new ways for obtaining more realistic molecular systems description through exact and approximate solutions of quantum wave equations. Against this backdrop, Ebomwonyi et al. [16] developed the Coulomb-Hulthén-Pöschl-Teller (CHPT) interaction potential model, a novel hybrid interaction potential formed from the combination of the Coulomb potential, the Hulthén potential and the Pöschl-Teller potential. The long-range interaction is accounted for by the Coulomb potential, while the Hulthén and Pöschl-Teller potentials describe the short-range function of the molecular force field. The CHPT interaction potential has a well-defined confining structure that can support discreet bound states, and thus provides an efficient framework for the description of molecular vibrational motion. Employing the parametric Nikiforov-Uvarov (NU) technique, Ebomwonyi et al. [16] solved the radial Schrödinger equation to obtain expressions for the energy eigenvalues, and wavefunctions related to the CHPT potential; and thereafter, they investigated the thermodynamic properties of the ground state of CO. Their results showed that the CHPT potential gives a physically meaningful description of the thermodynamic behavior of CO and demonstrated the CHPT potential model as a viable option for molecular studies.

In spite of the significant contribution of Ebomwonyi et al. [16], their work was primarily on thermodynamic description of CO, rather than the basic issue of molecular stability. Even though the thermodynamic properties were obtained from the energy spectrum of CHPT, the study never explicitly investigated bound state spectrum structure that controls molecular confinement and dissociation. Particularly, key stability indicators like the evolution of energy eigenvalues with screening parameter, the dissociation threshold, and the domain of the stability of CO were not examined. All these quantities are of significant physical importance because they give a direct understanding of the conditions whereby a molecular system remains bound or unbound. Advances in studies of quantum stability have shown that bound state eigenvalues behavior contains information rich in phase transition-like phenomena in molecular systems. Energy level evolution, dissociation energies, bound state density, and critical coupling parameters have been analyzed and

successfully employed in the characterization of stability regions in atoms, molecules and condensed matter systems [17-19]. Deeper physical insights are offered from such approaches than purely thermodynamic studies because they probe the quantum structure responsible for confinement directly. Also, stability analysis provides a connection between molecular spectroscopy and quantum dynamics by linking measureable spectroscopic quantities with the fundamental interaction potential.

This study is therefore aimed at extending the thermodynamic framework established by Ebomwonyi et al. [16] to an extensive quantum mechanical description of molecular stability. It specifically investigates the stability behavior of CO under the CHPT potential through a detailed bound state energy spectrum analysis. Also, the CHPT potential was solved previously using the NU method, with the resulting energy spectrum employed to investigate the thermodynamic properties of CO. Though, NU method has been widely used in quantum mechanics, the Formula Method gives an alternative analytical framework for solving the Schrödinger equation, offering a more compact derivation and still producing the same bound state energy spectrum for the CHPT potential.

2.0 Methodology

2.1 The CHPT Potential Model

The CHPT potential model is given as [16] and it was applied in solving the radial Schrödinger equation.

$$V(r) = -\frac{A}{r} + \frac{Be^{-2\alpha r}}{1-e^{-2\alpha r}} - \frac{4Ce^{-2\alpha r}}{(1-e^{-2\alpha r})^2} \ , \tag{1}$$

where α is the screening parameter, A,B,C are potential strengths.

2.2 The Formula Method

The second-order differential equation is of the form

$$\psi_n^{\parallel}(s) + \left[\frac{\lambda_1 - \lambda_2 s}{s(1-\lambda_3 s)}\right]\psi_n^{|}(s) + \left[\frac{A_1 s^2 + A_2 s - A_3}{s(1-\lambda_3 s)}\right]\psi_n(s) = 0 \ . \tag{2}$$

The condition for energy equation with the Formula Method is given as [20]

$$\alpha_1^2 = \left[\frac{\alpha_0^2 - \alpha_1^2 - [\frac{1-2n}{2} - \frac{1}{2\lambda_3}(\lambda_2 - \sqrt{(\lambda_3-\lambda_2)^2 - 4K_1})]^2}{2[\frac{1-2n}{2} - \frac{1}{2\lambda_3}\left(\lambda_2 - \sqrt{(\lambda_3-\lambda_2)^2-4K_1}\right)]}\right]^2 , \tag{3}$$

where

$$\alpha_0^2 = \frac{(1-\lambda_1) + \sqrt{(1-\lambda_1)^2 - 4K_3})}{2} \tag{4}$$

and

$$\alpha_1^2 = \frac{1}{2} + \frac{\lambda_1}{2} - \frac{\lambda_2}{2\lambda_3} + \sqrt{\left(\frac{1}{2} + \frac{\lambda_1}{2} - \frac{\lambda_2}{2\lambda_3}\right)^2 - \frac{K_1}{\lambda_3^2} - \frac{K_2}{\lambda_2} - K_3} \ , \tag{5}$$

2.3 The Radial Schrödinger Equation

The radial Schrödinger equation presented in Eq. (6) is used with the CHPT potential to obtain bound state solutions.

$$\frac{d^2}{dr^2} + \left[\frac{2\mu}{\hbar^2}\left\{E - V(r) - \frac{L\hbar^2}{2\mu r^2}\right\}\right]R(r) = 0, \tag{6}$$

where

$$L = l(l+1) \tag{7}$$

is the orbital angular momentum and μ is reduced mass.

In order to deal with the centrifugal term $\frac{1}{r^2}$ in Eq. (6), we employ the recommended approximation scheme [21]

$$\frac{1}{r^2} \approx \frac{4\alpha^2 e^{-2\alpha r}}{(1-e^{-2\alpha r})^2}. \quad (8)$$

This approximation scheme preserves the correct asymptotic behaviour of the centrifugal barrier while allowing the radial Schrödinger equation to be solved analytically. Previous studies [22,23,24] have demonstrated that this approximation provides highly accurate bound-state energies for weak and moderate screening regimes, particularly for Hulthén-, Yukawa-, Manning–Rosen- and Pöschl–Teller-type potentials. Since the present investigation considers normalized screening parameters within the interval $\alpha = 0.05 - 0.20$, the approximation remains valid throughout the investigated parameter range and is therefore appropriate for the present stability analysis.

Substituting Eq. (1) into Eq. (6) and defining a transformational variable $s = e^{-2\alpha r}$, Eq. (6) becomes

$$\left(\frac{d^2}{ds^2} + \frac{(1-s)}{s(1-s)}\frac{d}{ds}\right) + \frac{1}{s^2(1-s)^2}[K_1 s^2 + K_2 s + K_3]R(r) = 0 \quad (9)$$

where

$$K_1 = \frac{1}{2\alpha^2\hbar^2}(\mu E - 2\mu A\alpha + \mu B) \quad (10)$$

$$K_2 = \frac{1}{2\alpha^2\hbar^2}(-2\mu E + 2\mu A\alpha - \mu B + 4\mu C) \quad (11)$$

$$K_3 = \frac{\mu}{2\alpha^2\hbar^2}\left(E - \frac{2\alpha^2\hbar^2 L}{\mu}\right). \quad (12)$$

Comparing Eq. (9) with Eq. (2), we obtain

$$\lambda_1 = \lambda_2 = \lambda_3 = 1, \quad (13)$$

$$\alpha_0^2 = -\frac{\mu E}{2\alpha^2\hbar^2} + L, \quad (14)$$

$$\alpha_1^2 = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\mu C}{\alpha^2\hbar^2} + L}. \quad (15)$$

Now, if we substitute Eqs. (13), (14) and (15) into Eq. (3), we obtain the energy expression of the CHPT potential as

$$E_{nl} = \frac{2\alpha^2\hbar^2}{\mu}\left\{L - \frac{A\mu}{\alpha\hbar^2} - \left[\frac{\frac{2\mu}{\alpha^2\hbar^2}\left(C + \frac{\alpha A}{2} - \frac{B}{4} - \frac{\alpha^2\hbar^2(2L+1)^2}{4\mu} - \frac{\alpha^2\hbar^2 n(n+1)}{2\mu} - \left(n+\frac{1}{2}\right)\sqrt{(2l+1)^2 - \frac{8\mu C}{\alpha^2\hbar^2}}\right]}{1 + 2n + \sqrt{(2l+1)^2 - \frac{8\mu C}{\alpha^2\hbar^2}}}\right]^2\right\}, \quad (16)$$

where *n* is the principal quantum number.

2.4 Percentage Stability Change

To quantify the extent to which each state preserves its binding strength under increasing screening, we calculate the percentage stability loss using Eq. (17).

$$Percentage\ stability\ change = \left(\frac{|E_{\alpha i}| - |E_{\alpha f}|}{|E_{\alpha i}|}\right) X100, \quad (17)$$

where $E_{\alpha i}$ and $E_{\alpha f}$ are the energy eigenvalues corresponding to the initial screening parameter and final energy eigenvalues respectively.

3.0 Results and Discussion

Table 1: CHPT energy spectra for *2p*, *3p* and *3d* quantum states for α value of 0.05 to 0.20 ($\hbar = \mu = 1$)

Quantum state	α value	B = A = -C = 1	B = 2A = -2C = 2	2B = A = -2C =2
2p	0.05	-1.6061	-2.2704	-1.6462
	0.10	-1.5987	-2.2325	-1.6858
	0.15	-1.5462	-2.1436	-1.6884
	0.20	-1.4547	-2.0112	-1.6613
3p	0.05	-1.6764	-2.3299	-1.7172
	0.10	-1.7479	-2.3652	-1.8365
	0.15	-1.7820	-2.3627	-1.9257
	0.20	-1.7838	-2.3281	-1.9890
3d	0.05	-1.5549	-2.2112	-1.5959
	0.10	-1.4043	-2.0084	-1.4970
	0.15	-1.1366	-1.6731	-1.2963
	0.20	-.0.7813	-1.2403	-1.0251

In Table 1, the numerical spectra of CO under the influence of the CHPT potential is reported and it consists purely of eigenvalues that are negative for all the quantum states over the screening range from 0.05 to 0.20 used in this study. This range supports the regime where stable molecular configurations are preserved; that is, the selected screening interval represents the weak-to-moderate screening regime commonly adopted in analytical molecular potential studies. Within this interval the interaction remains sufficiently long-ranged to preserve bound states while allowing the effect of screening on molecular stability to be investigated systematically. The negative values as shown in Table 1 are crucial in the determination of stability because they are associated with bound states, suggesting the CO remains confined within the CHPT potential well for all the parameter values considered. There is absence of positive eigenvalues in the entire spectrum, indicating that none of the states under consideration crosses the dissociation threshold; providing compelling evidence that the CHPT interaction potential supports stable molecular confinement.

Figure 1 presents the eigenvalue spectrum and stability phase diagram obtained from the CHPT potential. The variations of the bound state energies for the *2p*, *3p* and *3d* quantum states as a function of the screening parameter α, and together with the dissociation threshold is depicted in the figure. The dotted horizontal line (E = 0) differentiates the stable bound state region (E < 0) from the unbound region (E > 0) and thus provides a direct quantum mechanical benchmark for assessing stability in molecules. A distinct characteristic for the different quantum states is revealed from the Figure 1. The most negative eigenvalues were exhibited by the *3p* state and is farthest below the dissociation threshold. The energy decreases from -1.6764 at α = 0.05 to -1.7838 at α = 0.20, indicating improved localization as the screening parameter increases; and suggesting that the CHPT potential creates a particularly favourable environment for the *3p* state. As such, the *3p* state represents the most stable quantum state within the CHPT spectrum. Different response to screening is exhibited by the *2p* and *3p* quantum states. Both states move progressively towards the continuum threshold as the screening parameter increases. The energies in the *2p* state move from -1.6061 to -1.4547, while that of the *3d* state moves significantly from -1.5549 to -0.7813. This swift upward movement of the eigenvalues indicates a reduction in the binding strength, resulting from enhanced screening of the attractive interaction. Physically, the effective interaction range is shortened when α is increased, thereby reducing the confining capability of the molecular potential. The *3d* state exhibits the greatest sensitivity to screening among the investigated quantum states and therefore represents the weakest bound state in the spectrum. Nonetheless, the *3d* spectrum remains greatly below the dissociation threshold, even at the largest screening parameter considered. Although, screening weakens confinement, it is insufficient to induce CO dissociation within the studied parameter range. The separation between each

eigenvalue trajectory and the dissociation threshold gives a quantitative measure of stability. Quantum states located farther below the threshold value possesses larger binding energies and therefore strong resistance to external perturbations. As a result of this criterion, the ordering of the stability within the investigated states is *3p* > *2p* > *3d*. This arrangement remains the same throughout the studies screening interval, indicating that the CHPT interaction potential retains the relative stability hierarchy of the quantum states. Also worthy of note from Figure 1 is the steady progression of the energy spectrum. There was no observed discontinuity, level crossings or sudden spectra transitions. This kind of behavior is characteristic of stable quantum systems and proves the reliability of the CHPT potential model in the description of molecular interactions.

The changes observed in molecular stability under increasing screening have important implications for the spectroscopic behaviour of the CO molecule because vibrational and rotational spectra arise directly from transitions between discrete bound state energy levels. The persistence of negative energy eigenvalues throughout the investigated screening interval indicates that the bound vibrational states remain intact, implying that the characteristic infrared and microwave spectra of CO would continue to be observable without evidence of dissociation. However, the gradual reduction in the binding strength of the *2p* and *3d* states suggests a compression of the molecular potential well, which would modify the spacing between adjacent energy levels and could lead to small shifts in the corresponding transition frequencies. Such changes may appear experimentally as slight variations in spectral line positions or transition energies. Conversely, the enhanced stability observed for the *3p* state indicates stronger localization of the molecular wavefunction, which is expected to preserve the integrity of the associated bound-state transitions against screening-induced perturbations. The absence of level crossing throughout the investigated screening range further implies that no abrupt spectral rearrangement or disappearance of transition lines is expected, thereby confirming the robustness of the CHPT potential in describing the spectroscopic stability of the CO molecule.

Table 2: Percentage stability change in CHPT potential for 2p, *3p* and *3d* states under increasing screening

State	Stability change (%)
2p	9.43
3p	-6.40
3d	49.75

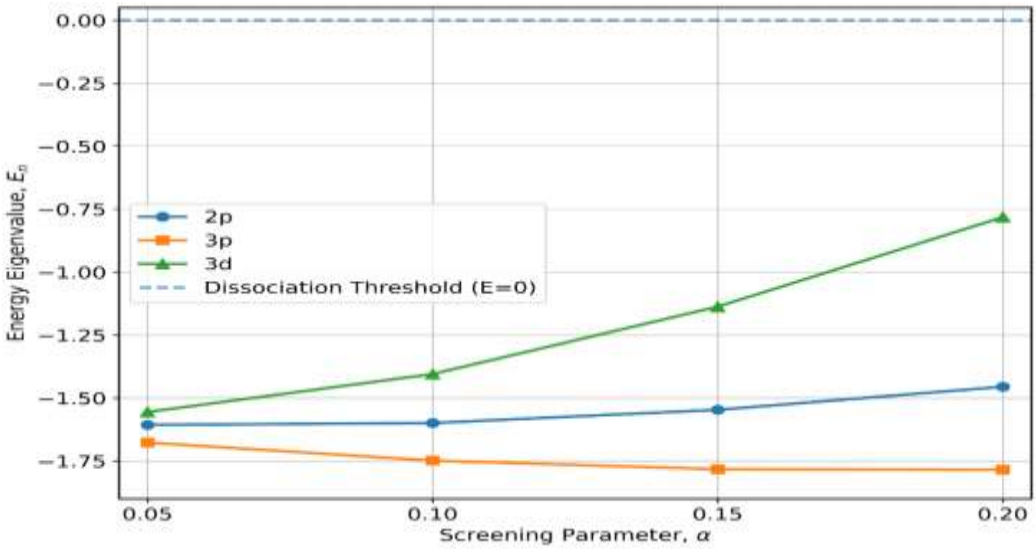

Figure 1: CHPT potential eigenvalue spectrum and stability phase analysis of CO

Table 2 shows the percentage stability change for the *2p*, *3p* and *3d* quantum states within the investigated screening interval calculated from Eq. (17). It is observed that the 3p state exhibits a negative percentage stability change, indicating that the magnitude of its negative energy eigenvalue increases with screening. This corresponds to enhanced molecular binding and therefore represents a stability gain rather than a loss. In contrast, the positive stability-change values observed for the 2p and 3d states indicate a reduction in binding strength as screening increases.

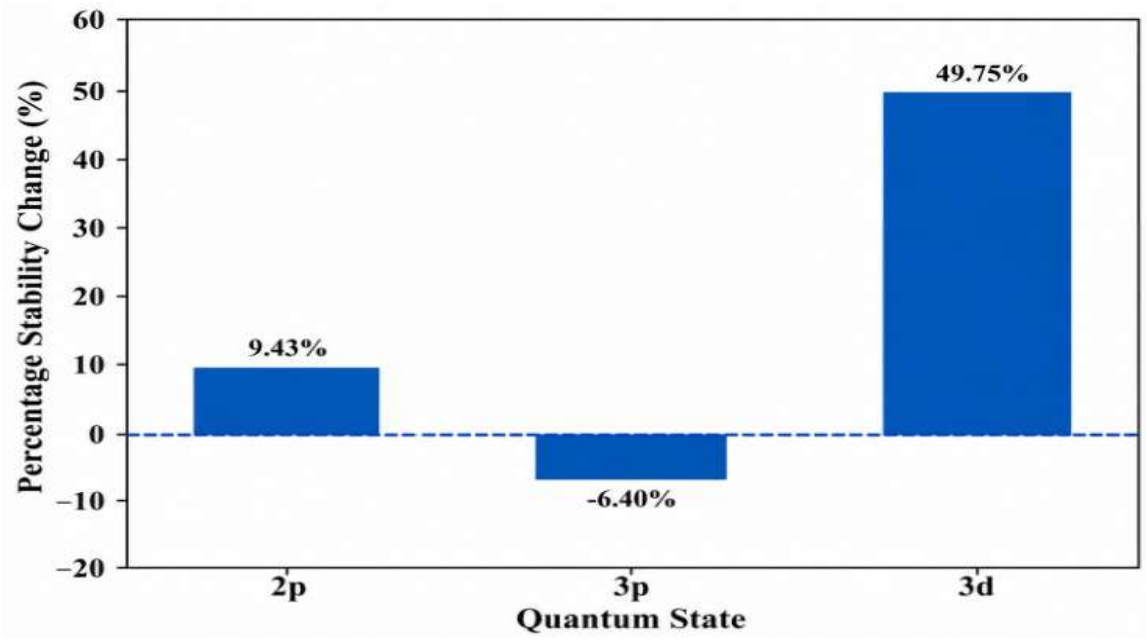

Figure 2: CHPT robustness plot of CO under screening

The percentage stability change of the *2p, 3p* and *3d* quantum states as screening parameter increases from 0.05 to 0.20 is shown in Figure 2. This graph, unlike the stability phase graph in Figure 1 quantifies the extent to which each quantum state preserves its binding strength under increasing screening. That is, it serves as a measure of the resilience of the molecular spectrum to perturbations in the interaction range. The largest response to screening is shown by the *3d* state with a reduction in binding strength of almost 50%. This high variation indicates that the *3d* state is highly susceptible to changes in the effective interaction range and therefore gives the most screening-dependent component of the spectrum. Sensitivity of this magnitude suggests that the spatial distribution of the *3d* wavefunction is strongly influenced by changes in the screening environment, leading to a pronounced alteration of its confinement characteristics. A stability change of almost 10% was displayed by the *2p* state, indicating that its binding characteristics remain relatively insensitive to variations in the screening parameter. The underlying confinement mechanism remains largely preserved throughout the investigation interval, an indication of the modest change observed for the *2p* state; and reflecting a greater degree of structural resilience within the CHPT potential framework. A striking feature of the robustness analysis is the negative stability change value observed for the *3p* state. It shows that the magnitude of the binding energy increases with increasing screening, which corresponds to a net gain in stability rather than a loss. This behavior shows the non-uniform response of several quantum states to the same perturbation, and portrays the flexibility of the CHPT potential in accommodating state dependent confinement effects. Therefore, the bound state robustness plot reveals that the influence of screening is hugely state selective. Whereas some states experience great reduction in binding strength, others remain significantly unaffected or exhibit enlarged confinement. This response of state-dependent provides more insight into the internal structure of CHPT CO spectrum, demonstrating that stability cannot be fully characterized mainly by the sign of the eigenvalues. Rather, the extent at which binding strength vary under perturbations offers an additional quantitative criterion for assessing the resilience of molecular states.

Physical Significance of the 3p State Stability Gain

An interesting feature of the CHPT potential is the enhanced stability exhibited by the 3p quantum state as the screening parameter increases. Unlike the 2p and 3d states, whose energy eigenvalues move progressively towards the dissociation threshold, the 3p state becomes increasingly negative, indicating a gain in binding strength. This behaviour suggests that the combined Coulomb–Hulthén–Pöschl–Teller interaction produces a favourable balance between long-range attraction and short-range confinement for this particular quantum state. As screening increases, the effective interaction suppresses the less attractive long-range component while simultaneously enhancing the contribution of the short-range Hulthén and Pöschl–Teller terms. The spatial distribution of the 3p wavefunction allows it to benefit more effectively from this modified potential landscape, resulting in stronger localization within the potential well. Consequently, the probability of finding the molecule in a tightly confined bound state increases, thereby improving its resistance to screening-induced perturbations.

The enhanced localization of the 3p state also has important spectroscopic implications. Since transition frequencies depend on the spacing between bound-state energy levels, a more strongly bound 3p state is expected to exhibit greater robustness against environmental perturbations, preserving the integrity of the associated vibrational and rotational transitions. This state-selective stabilization demonstrates that molecular stability under the CHPT potential is governed not only by the magnitude of the screening parameter but also by the interplay between the hybrid interaction potential and the quantum-state characteristics. Such behaviour highlights the ability of the CHPT potential to capture confinement effects that are not readily reproduced by conventional single-component interaction potentials.

Table 3: Comparative characteristics of CO stability under different molecular potentials

Potential Model/Ref.	Method	Key stability observation	Stability evidence
Shifted Deng-Fan potential [18]	NU Method	Suggests stable bound states with a monotonic increase of eigenvalues with quantum number as spectra ordering is preserved	Stable discrete spectrum and preserved state ordering
Generalized Morse potential [22]	Shape invariance/SUSY Method	Exhibit a stable vibrational spectrum with improved treatment of anharmonicity compared to the Morse potential	Stable anharmonic vibrational levels
CHPT potential (this study)	Formula Method	It shows that negative bound state spectrum remains below the dissociation threshold all through the investigated screening interval, with no level crossing observed. The percentage stability loss and stability phase-diagram analysis show persistent quantum stability	Negative bound state spectrum, stability phase diagram, percentage stability loss analysis and no level crossing

A summary of the stability analysis predicted by different analytical interaction potentials reported in the literature alongside the CHPT potential used in this study is presented in Table 3. In spite of the different mathematical formulations, all the interaction potential models presented predict a negative discrete bound state spectrum, establishing the excellent quantum stability of CO. This agreement shows that the presence of stable vibrational states is a strong physical characteristic that is highly independent of the particular analytical form of the interaction potential. The major difference of the present study lies on how molecular stability is analyzed. The Generalized Morse and the Shifted Deng-Fan potential both focused majorly on obtaining accurate rotational-vibrational energy spectra and thermodynamic properties. The present study emphasizes the stability characteristic of CHPT potential by evolution of the bound state spectrum in relation to the screening parameter, the stability phase-diagram in relation to the dissociation threshold, and the percentage stability loss analysis. A direct quantum mechanical interpretation of the persistence of molecular confinement as the interaction becomes increasingly screened is provided by these complementary indicators. Though, the CHPT potential and the Shifted Deng-Fan potential differ greatly in their analytical framework, they both predict stable bound configurations for CO and retain the quantum states ordering all through the parameter range investigated. Furthermore, the absence of level crossing in the present CHPT analysis further affirms the robustness of the molecular binding predicted by the CHPT interaction potential. Therefore, the combination of Coulomb, Hulthén and Pöschl-Teller potentials into a single interaction potential provides a different framework for investigating molecular stability and at the same time remaining consistent with the general stability behavior as reported by other established diatomic molecular potentials.

Table 4: Comparison of bound state energies predicted by different interaction potentials for CO

Quantum state	Yukawa potential [23]	MGESCY potential [23]	α value	CHPT (This study)	α value
2p	-1.7720 -1.7456	-1.7720 -1.7456	0.01 0.03	-1.6061 -1.5987 -1.5462 -1.4547	0.05 0.10 0.15 0.20
3p	-1.1471 -1.1448	-1.1471 -1.1448	0.01 0.03	-1.6764 -1.7479 -1.7820 -1.7838	0.05 0.10 0.15 0.20
3d	-0.8048 -0.8184	-0.8048 -0.8184	0.01 0.03	-1.5549 -1.4043 -1.1366 -0.7813	0.05 0.10 0.15 0.20

MGESCY (More Generalized Exponential Screened Coulomb Potential plus Yukawa Potential).

Table 4 compares the bound-state energy eigenvalues of CO predicted by the CHPT potential with those obtained using the Yukawa and MGESCY potentials [25]. Although the screening parameter ranges differ between the studies, all three interaction potentials consistently predict **negative energy eigenvalues** for the *2p*, *3p* and *3d* quantum states, confirming that the investigated states remain below the dissociation threshold and therefore represent stable bound molecular configurations. The CHPT potential predicts a gradual reduction in the binding strength of the *2p* and *3d* states with increasing screening, while the *3p* state exhibits enhanced binding, indicating a state-dependent response to screening. In contrast, the Yukawa and MGESCY potentials predict relatively small changes in the corresponding eigenvalues within their investigated screening interval, suggesting weaker sensitivity to screening. Despite these quantitative differences, the agreement among the three potentials in predicting stable bound states and preserving the ordering of the quantum states demonstrates the physical consistency and reliability of the CHPT potential for describing the quantum stability of the carbon monoxide molecule.

The present stability analysis complements the previous thermodynamic investigation of the CHPT potential [16] because thermodynamic properties are determined by the underlying energy spectrum through the partition function. As screening weakens the binding of the *2p* and *3d* states, the reduced energy separation between bound levels is expected to increase the thermal accessibility of excited states, which may lead to corresponding increases in entropy and modifications of the heat capacity. Conversely, the enhanced stability of the *3p* state implies stronger localization and a reduced probability of thermal excitation from that level, thereby contributing to greater thermodynamic stability. These findings demonstrate that the observed state-dependent stability changes provide the quantum mechanical basis for the thermodynamic behaviour previously reported for the CO molecule.

Conclusion

A comprehensive quantum stability analysis of CO within the CHPT potential framework has been presented in this study. This was done by re-examining the bound state eigenvalue spectrum earlier reported with the analysis demonstrating that all investigated eigenvalues remain below the dissociation threshold all through the considered screening interval; confirming the existence of

stable bound states and strong confining capability of the CHPT interaction potential. It was further established by the stability phase diagram that no critical screening parameter is attained within the investigated range, suggesting that CO remains resistant to screening-induced dissociation. Furthermore, it was also revealed by the bound state robustness analysis that there is a state-dependent response to screening, with enhanced confinement exhibited by the *3p* state, the *2p* state showing on a marginal reduction in stability, and the *3d* state displaying the highest sensitivity to screening perturbations. These results show that molecular stability is not only governed by negative eigenvalues, but also by the robustness of the bound state spectrum versus variations in the interaction parameters. The enhanced stability of the *3p* state under increasing screening constitutes a distinctive prediction of the CHPT potential, revealing that hybrid interaction potentials can produce state-dependent confinement effects that are not captured by conventional screened molecular potentials. This finding provides new insight into the quantum mechanical mechanisms governing molecular stability and suggests that hybrid potentials may offer improved descriptions of selective bound state localization in diatomic molecules.

References

[1] Dong, S.H. (2007). Factorization Method in Quantum Mechanics. Springer, Amsterdam, 150-155. https://doi.org/10.1007/978-1-4020-5796-0.

[2] Griffiths, D.J. and Schroeter, D.F. (2018). Introduction to Quantum Mechanics. Cambridge University Press, Cambridge. https://doi.org/10.1017/9781316995433.

[3] Flügge, S. (1999). Practical Quantum Mechanics II. Springer Berlin, Heidelberg. https://doi.org/10.1007/978-3-642-65114-4.

[4] Herzberg, G. (1950). Molecular Spectra and Molecular Structure. I. Spectra of Diatomic Molecules, Van Nostrand, Princeton, 194-204.

[5] Huber, K.P. and Herzberg, G. (1979). Molecular Spectra and Molecular Structure: IV Constants of Diatomic Molecules. Van Nostrand Reinhold Company, New York. http://dx.doi.org/10.1007/978-1-4757-0961-2.

[6] Lefebvre-Brion, R. and Field, R.W. (2004). The Spectra and Dynamics of Diatomic Molecules: Revised and Enlarged Edition. Elsevier Academic Press. https://doi.org/10.1016/b978-0-12-441455-6.X5000-8.

[7] Morse, P.M. (1929). Diatomic molecules according to the wave mechanics. *Physical Review*, 34(1), 57–64.

[8] Kratzer, A. (1920). Die ultraroten Rotationsspektren der Halogenwasserstoffe. *Zeitschrift für Physik*, 3, 289–307.

[9] Hulthén, L. (1942). On the theory of screened Coulomb fields. *Arkiv för Matematik, Astronomi och Fysik, 28A*, 1–28.

[10] Manning, M.F. and Rosen, N. (1933). A potential function for the vibrations of diatomic molecules. *Physical Review*, 44(11), 953–959.

[11] Deng, Z.H. and Fan, Y.P. (1957). A potential function of diatomic molecules. *Shandong University Journal*, 7, 162–168.

[12] Pöschl, G. and Teller, E. (1933). Bemerkungen zur Quantenmechanik des anharmonischen Oszillators. *Zeitschrift für Physik*, 83, 143–151.

[13] Ikhdair, S.M. and Sever, R. (2010). Exact and approximate solutions of relativistic wave equations for combined interaction potentials. *Central European Journal of Physics*, 8(5), 652–666.

[14] Onate, C.A. and Ojonubah, J.O. (2016). Thermodynamic properties of systems under generalized Pöschl–Teller and hyperbolical potentials. *Journal of Theoretical Physics and Applications*, 10, 21–31.

[15] Ita, B.I. and Ikot, A.N. (2012). Solutions of the Schrödinger equation for inversely quadratic Hellmann potentials. *Pramana – Journal of Physics*, 79(3), 345–357.
[16] Ebomwonyi, O., Onate, C.A., Ekong, S.A. and Onyeaju, M.C. (2019). Thermodynamic properties for the carbon monoxide molecule under the influence of the Coulomb–Hulthén–Pöschl–Teller potential. *Journal of Science and Technology Research*, 1(1), 122–136.
[17] Dong, S.H. and Cruz-Irisson, M. (2012). Energy spectrum and thermodynamic properties of molecular systems under modified Rosen–Morse interactions. *Journal of Mathematical Chemistry*, 50, 881–893.
[18] Oyewumi, K.J., Falaye, B.J., Onate, C.A., Oluwadare, O.J. and Yahya, W.A. (2013). Thermodynamic properties and approximate solutions of the Schrödinger equation with shifted Deng–Fan potential model. *Molecular Physics, 112*(1), 127–141.
[19] Falaye, B.J., Oyewumi, K.J., Ibrahim, T.T. and Punyasena, M.A. (2013). Bound-state solutions and thermodynamic properties of quantum systems under combined potentials. *Chinese Physics B, 22*(11), 110301.
[20] Falaye, B.J., Ikhdair, S.M. and Hamzavi, M. (2015). Formula method for bound state problems. *Few-Body System*, 56, 63-78.
[21] Greene, R.L. and Aldrich, C. (1976). Variational wave functions for a screened Coulomb potential. *Physical Review A*, 14, 2363-2366.
[22] Wei, G.F. and Dong, S.H. (2008). Approximately analytical solutions of the Manning–Rosen potential with the spin–orbit coupling term and spin symmetry. *Physics Letters A*, 373, 49-53.
[23] Dhahbi, A. and Landolsi, A.A. (2022). The Klein-Gordon equation with equal scalar and vector Bargmann potentials in *D* dimensions. *Results in Physics*, 33, 105143.
[24] Okon, I., Onate, C., Omugbe, E., Okorie, U., Antia, A., Onyeaju, M., Wen-Li, C. and Araujo, J. (2022). Approximate solutions, thermal properties and superstatistics solutions to Schrödinger equation. Advances in High Energy Physics, 2022, 5178247. https://doi.org/10.1155/2022/5178247.
[25] Ita, B.I., Louis, H., Akakuru, O.U. Magu, T.O., Joseph, I., Tchoua, P., Amos, P.I., Effiong, I. and Nzeata, N.A. (2018). Bound state solutions of the Schrödinger equation for the More Generalized Exponential Screened Coulomb Potential plus Yukawa (MGESCY) potential using Nikiforov-Uvarov method. *Journal of Quantum Information Science*, 8, 24-45.