Journal of TNAMP 25 (2026) 105-123

Transactions of
The Nigerian Association of
Mathematical Physics
Journal homepage: https://nampjournals.org.ng

OPTIMAL CONTROL MATHEMATICAL MODEL OF MALARIA AND TYPHOID FEVER CO-INFECTION DYNAMICS WITH ENVIRONMENTAL DEPENDENCY

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ARTICLE INFO

Article history:
Received xxxxx
Revised xxxxx
Accepted xxxxx
Available online xxxxx

Keywords:
Malaria- typhoid co-infection,
Basic Reproduction Number,
Optimal Control,
Environmental dependency,
Sensitivity analysis.

ABSTRACT

This study develops an optimal control mathematical model for malaria-typhoid co-infection dynamics with environmental dependency. The model incorporates, human populations, mosquito breeding vectors environments, Salmonella typhi bacteria reservoirs, and multiple control strategies within a unified framework. A system of nonlinear ordinary differential equations is used to describe the transmission dynamics. The Disease-free Equilibrium (DFE), Endemic Equilibrium Point (EEP), and basic reproduction numbers (R_0) are derived and analysed. Stability analysis shows that DFE is locally asymptotically stable when $R_0 < 1$ and unstable when $R_0 > 1$. Sensitivity analysis identifies the key parameters that have the greatest influence on the disease transmission and persistence. Pontryagin's Maximum Principle is applied to determine the optimal control strategies by minimizing the number of infected individuals, environmental reservoirs diseases, and the implementation costs of the control measures. Numerical simulations show that the combined implementation of treatment and environmental sanitation effectively reduces malaria-typhoid co-infection and disease burden. The findings emphasize the importance of integrating treatment with environmental management to support optimal and sustainable public health interventions for controlling malaria-typhoid co-infection and also provide useful insights for policymakers.

1. Introduction

Malaria and Typhoid fever remain importance public health challenges in many developing countries, particularly in Sub-Saharan Africa, where environmental and socio-Economic conditions facilitate diseases transmission. Malaria is a mosquito-borne disease transmitted by infected female Anopheles mosquitoes, while typhoid fever is a bacterial infection caused by salmonella enterica serover typhi and transmitted primary through contaminated food and water.

The co-existence of these diseases in endemic regions often leads to co-infection, complicating diagnosis, treatment, and disease management [1] and Outbreaks of infectious diseases [2].

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https://doi.org/10.60787/tnamp.v25.735

Environmental factors significantly influence the transmission dynamics of both diseases [3]. We developed an optimal mathematical model for malaria-typhoid fever co-infection dynamics incorporating Environmental Dependency. However, the intervention strategies aimed at reducing the disease burden was not investigated. Motivated by this limitation, we presented four time-dependent control measures. Using optimal control theory and Pontryagin's Maximum Principle [4], the study evaluates these interventions in minimizing disease prevalence and environmental contamination, and the implementation costs of the control measures, to evaluate integrated intervention strategies [4]. Previous studies have model single infections and malaria-typhoid co-infection dynamics; few have explicitly captured the interactions among the human population, mosquito breeding vectors, Salmonella typhi bacteria, and environmental factors within a unified mathematical framework [3]. We extend prior work by developing a novel compartmental model that incorporates both direct and indirect transmission pathways for malaria and typhoid fever [5]. The model explicitly incorporates environmental reservoirs, including contaminated water sources and mosquito breeding sites to investigate how these reservoirs influence disease dynamics and co-infection dynamics [6]. The remainder of this paper is structured as follows: Section 2 presents the model formulation, including all compartments and transmission pathways, Positivity and boundedness, DFE and EEP, Basic Reproduction numbers and sensitivity analysis. Section 3 present optimal control theory using four control variable and the Pontryagin's Maximum Principle to evaluate intervention strategies [4]. Section 4 present graphical results for the optimal control simulation, while section 5 present discussions of numerical results and conclusion.

2.0 Material and Methods

2.1 Model Formulation

$$N_h = \quad I_t(t) + I \quad - I_{mt}(t) + R_t(t) + I \quad \mu_h + \quad \quad _{mt}(\quad \rho_t R_t \quad) I_v(t)$$

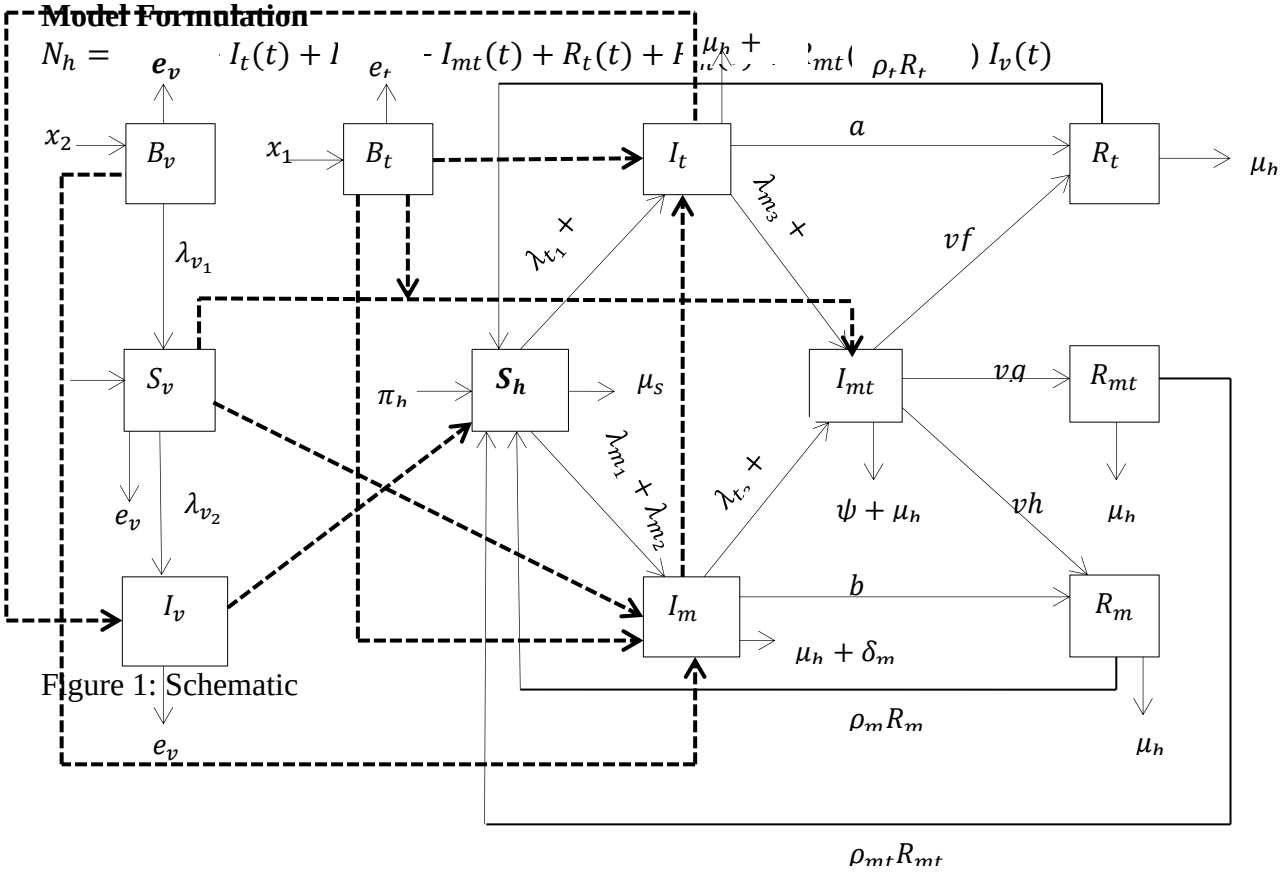

Figure 1: Schematic

Flow Diagram for Malaria and Typhoid Fever Co-Infection

2.2 Transmission Dynamics

a) Typhoid fever infection to susceptible Humans

Direct (person-to-person) and indirect (environment) transmission

$$\lambda_{t_1} = \frac{\beta_{t1}\ (\ I_t + \mathrm{n}_t I_{mt}\ )}{N_h}, \quad \lambda_{t_2} = \frac{\beta_{t2}\ \ B_t}{B_t + \ K_t}$$

b) **Malaria Infection to Susceptible Humans**

Direct (person-to-person) and indirect (environment) transmission

$$\lambda_{m_1} = \frac{\beta_{m1}\ (\ I_m + \mathrm{n}_m I_{mt}\ )}{N_h}, \ \lambda_{m_2} = \beta_{m_2}\left(\frac{B_v}{B_v + \ K_v}\right) I_m$$

2.3 Cross-Transmission: Typhoid and Malaria Co-Infection

c) Typhoid to Malaria-infected individual

$$\lambda_{t_3} = \frac{\beta_{t3}(\ I_t + \mathrm{n}_t I_{mt}\ )}{N_h}, \ \lambda_{t_4} = \frac{\beta_{t4}\ \ B_t}{B_t + \ K_t}$$

d) Malaria to Typhoid-infected individual

$$\lambda_{m_3} = \frac{\beta_{m3}\ (r\ I_m + \mathrm{n}_m I_{mt}\ )}{N_h}, \ \lambda_{m_4} = \beta_{m4}\ (\frac{B_v}{B_v + \ K_v}) I_m$$

2.4 Environmental Contamination

Contamination Dynamics in the environmental reservoirs are governed by logistic growth, shedding and decay:

Salmonella Typhi:

$$\frac{dB_t}{dt} = g_t B_t(1\ - \frac{B_t}{K_t}) + J_t I_t + \theta_t I_{mt} - e_t B_t$$

Aquatic-stage (breeding) vectors:

$$\frac{dB_v}{dt} = g_v B_v(1 - \frac{B_v}{K_v}) + J_v I_v + \theta_v I_{mt} - e_v B_v$$

$$\frac{dS_h}{dt} = \pi_h + \rho_t R_t + \rho_m R_m + \rho_{mt} R_{mt} - [\ \lambda_{t_1} + \lambda_{t_2} + \ \lambda_{m_1} + \ \lambda_{m_2}]S - \ \mu_s \mathrm{S}$$

$$\frac{dI_t}{dt} = (\ \lambda_{t_1} + \lambda_{t_2})\ S - \left(\ \lambda_{m_3} + \ \lambda_{m_4}\right) I_t - (\ \mu_h + \delta_T + a)\ I_t$$

$$\frac{dI_m}{dt} = (\ \lambda_{m_1} + \ \lambda_{m_2})\ S - \left(\ \lambda_{t_3} + \ \lambda_{t_4}\right) I_m - (\ \mu_h + \delta_m + b)\ I_m$$

$$\frac{dI_{mt}}{dt} = \left(\ \lambda_{t_3} + \ \lambda_{t_4}\right)\ I_m + \left(\ \lambda_{m_3} + \ \lambda_{m_4}\right) I_t - v\big((f + g + h)\big) I_{mt} - (\ \mu_h + \psi) I_{mt}$$

$$\frac{dR_t}{dt} = aI_T + v(f) I_{mt} - (\ \mu_h + \ \rho_t)\ R_t$$

$$\frac{dR_m}{dt} = bI_m + v(h) I_{mt} - (\ \mu_h + \ \rho_m)\ R_m \qquad (1)$$

$$\frac{dRm_t}{dt} = vg\ I_{mt} - (\ \mu_h + \ \rho_{mT}) R_{mt}$$

$$\frac{dB_t}{dt} = g_t B_t(1 - \frac{B_t}{K_t}) + J_t I_t + \theta_t I_{mt} - e_t B_t$$

$$\frac{dB_v}{dt} = g_v B_v(1 - \frac{B_v}{K_v}) + J_v I_v + \theta_v I_{mt} - e_v B_v$$

$\frac{dS_v}{dt} = \pi_v + \lambda_{v_1} B_v - (e_v + \lambda_{v_2}) S_v$

$\frac{dI_v}{dt} = \lambda_{v_2} S_v - e_v I_v$

Substituted into the above flow diagram are:

$x_1 = g_t B_t \left(1 - \frac{B_t}{k_t}\right) + J_t I_t + \theta_t I_{mt} - e_t B_t$, $x_2 = g_v B_v \left(1 - \frac{B_v}{k_v}\right) + J_v I_v + \theta_v I_{mt} - e_v B_v$

The Initial condition of the co-infection model are:

$S_h(0) > 0,\ I_t(0) \geq 0, I_m(0) \geq 0, I_{mt}(0) \geq 0, R_t(0) \geq 0, R_m(0) \geq 0, R_{mt}(0) \geq 0, B_t(0) \geq 0, B_v(0) \geq 0, S_v(0) \geq 0, I_v(0) \geq 0$

2.5 Positivity and Boundedness of the Malaria-typhoid fever co-infection Model.

For the model to be epidemiologically meaningful, it is important to examine the following in the closed set $\mathfrak{n}$ [3].

Theorem 1

The solution of the proposed model (1) is bounded

Proof

Let N_h denotes the number of total population for humans;

$X(t) = S_h + I_t + I_m + I_{mt} + R_t + R_m + R_{mt} + B_t + B_v + S_m + I_m$

$N_h(t) = S_h(t) + I_t(t) + I_m(t) + I_{mt}(t) + R_t(t) + R_m(t) + R_{mt}(t)$ (2)

$\frac{dN_h}{d_t} = \frac{dS_h}{d_t} + \frac{dI_T}{d_t} + \frac{dI_m}{d_t} + \frac{dI_t}{d_t} + \frac{dI_{mt}}{d_t} + \frac{dR_t}{d_t} + \frac{dRm}{d_t} + \frac{dR_{mt}}{d_t}$

$\frac{dN_h}{d_t} = \pi_h + P_t R_t + P_m R_m + P_{mt} R_{mt} - \mu_h N_h - [\sigma_t I_t + \sigma_m I_m + \psi \sigma_{mt} I_{mt}]$

Let $Z = P_t R_t + P_m R_m + P_{mt}$ and σ_t, σ_m, σ_{mt} represent infection induced death rate.

If there is no mortality of malaria and typhoid fever infection, we get

$\frac{dN_h}{d_t} \leq \pi_h + z - \mu_h N_h$

Thus, $\frac{dN_h}{d_t} + \mu_h N_h \leq \pi_h + z$ (3)

Integrating and simplifying both sides of the inequality, we obtain

$N_h(t) \leq \frac{\pi_h + z}{\mu_h} + N_h(0) - \frac{\pi_h + z}{\mu_h}) e^{\mu h t}$

$\lim_{n\to\infty} sup N_h(t) \leq \lim_{n\to\infty} \sup \left(\frac{\pi_h + z}{\mu_h} + (N_h(0) - \frac{\pi_h + z}{\mu_h}) e^{\mu h^t}\right) = \frac{\pi_h + z}{\mu_h}$

$\mathrm{n}_h = \left\{(S_h, I_t, I_m, I_{mt}, R_t . R_m, R_{mt}) \in R_+^7 : 0 \leq N_h \leq \frac{\pi_h + z}{\mu_h}\right\}$ (4)

From equation (1), we considers the environmental bacteria B_t:

$\frac{dB_t}{dt} = g_t B_t (1 - \frac{B_t}{K_t}) + J_t I_t + \theta_t I_{mt} - e_t B_t$

$\frac{dB_t}{dt} \leq g_t B_t (1 - \frac{B_t}{K_t})$

This is a logistic-type of equation

Thus, B_t is bounded above by carrying capacity K_t: $0 \leq B_t(t) \leq K_t$ for all $t \geq 0$ (5)

Similarly, we consider the vectors population at a given time t, given by

$N_v = B_v + S_v + I_v$

Differentiating equation (1) with respect to t, we have

$$\frac{dN_v}{d_t} = \frac{dS_v}{d_t} + \frac{dI_v}{d_t} + \frac{dB_v}{d_t} \quad (6)$$

This means that

$$\frac{d_{N_v}}{d_t} = \pi_v + \lambda_{v_1} B_v + \theta_v I_{mt} - \mu_v N_v$$

$$\frac{d_{N_v}}{d_t} \leq \pi_v + \lambda_{v_1} B_v + \theta_v I_{mt} - \mu_v N_v$$

Since $B_v \leq K_v$

Integrating and simplifying both sides of the inequality, we obtain

$$N3_v(t) \leq \frac{\pi_v + \lambda_{v_1} B_v}{\mu_v} + (N_v(o) - \frac{\pi_v + \lambda_{v_1} B_v}{\mu_v}) e^{-\mu v t}$$

Taking the limit, we have

$$\lim_{n\to\infty} sup N_v(t) \leq \frac{\pi_v + \lambda_{v_1} B_v}{\mu_v} : 0 \leq N_v(t) \leq \frac{\pi_v + \lambda_{v_1} B_v}{\mu_v}$$

Hence the total vector population N_v is bounded in the region;

$$\mathrm{n}_v = \left\{ (S_v, I_v, B_v) \in R_+^3 : 0 \leq N_v \leq \frac{\pi_v}{\mu_v} \right\} \quad (7)$$

Conclusively;

$$\mathrm{n} = \left\{ X = R_+^{11} : 0 \leq N_h \leq \frac{\pi_h + Z}{\mu_h}, 0 \leq B_t \leq K_t, 0 \leq \frac{\pi_v + \lambda_{v_1} B_v}{\mu_v} \right\} \quad (8)$$

Theorem 2

$S_h(0) > 0, I_t(0) > 0, I_m(0) > 0, I_{mt}(0) > 0, R_t(0) > 0, R_m(0) > 0, R_{mt}(0) > 0, B_t(0) > 0, B_v(0) > 0, S_h(0) > 0$ and $I_t(0) > 0$. Then the orbits$(s_h(t), I_t(t), I_m(t), I_{mt}(t), R_t(t), R_m(t), R_{mt}(t), B_t(t), B_v(t), S_v(t), I_v(t))$ of the model with positive basic conditions, will continue to be positive for all time *t > 0.*

Proof:

Let t_1= sup $\{t_1 > 0: S_h(0) > 0, I_t(0) > 0, I_m(0) > 0, I_{mt}(0) > 0, R_t(0) > 0, R_m(0) > 0, R_{mt}(0) > 0, B_t(0) > 0, B_v(0) > 0, S_h(0) > 0,\ I_t(0) > 0\}$

Consider the first equation of mode(1), given below as

$$\frac{ds_h(t)}{dt} = \pi_h + z - [\lambda_{t_1} + \lambda_{t_2} + \lambda_{m_1} + \lambda_{m_2}]S - \mu_s \mathrm{S}$$

Which can be re-expressed as:

$$\frac{d}{dt} = \left[S_h(t) exp\left\{ \mu_s \mathrm{S}^t + \int_0^t (\lambda_{t_1} + \lambda_{t_2} + \lambda_{m_1} + \lambda_{m_2}) dt \right\} \right] \geq$$

$$\pi_h + \mathrm{z}\ exp\left\{ \mu_s \mathrm{S}^t + \int_0^t (\lambda_{t_1} + \lambda_{t_2} + \lambda_{m_1} + \lambda_{m_2}) dt \right\}$$

$$S_h(t_1) exp\left\{ \mu t_1 + \int_0^{t_1} (\lambda_{t_1} + \lambda_{t_2} + \lambda_{m_1} + \lambda_{m_2}) dt \right\} - S_h(0)$$

$$\geq \int_0^{t1} \pi_h + \mathrm{z} \left[exp\left\{ \mu_s y + \int_0^y (\lambda_{t_1} + \lambda_{t_2} + \lambda_{m_1} + \lambda_{m_2}) dt \right\} \right] dy$$

So that

$$S_h(t_1) \geq S_h(0)exp\left[-\mu_s t_1 - \int_0^{t_1}(\lambda_{t_1} + \lambda_{t_2} + \lambda_{m_1} + \lambda_{m_2})dt\right] + \left[exp\left\{-\mu_s t_1 - \int_0^{t_1}(\lambda_{t_1} + \lambda_{t_2} + \lambda_{m_1} + \lambda_{m_2})dt\right\}\right] \times \int_0^{t_1} \pi_h + z\left[exp\{\mu_s y + \int_0^{y}(\lambda_{t_1} + \lambda_{t_2} + \lambda_{m_1} + \lambda_{m_2})dt\}\right]dy > 0 \quad (9)$$

Hence, $S_h(t) > 0, \forall\, t > 0$.

Similarly, following same process prove for others in equations (1). We have established positivity for all the state variables in model (1) for all time t.

2.6 Disease-Free Equilibrium Point (EEP) for Co-infection Model

At the DFE, there are no infections in the system, hence:

$$E_m^0 = (S_h^0,\ I_t^0, I_m^0, I_{mt}^0, R_t^0, R_m^0, R_{mt}^0, B_t^0, B_v^0, S_v^0, I_v^0) = (\frac{\pi_h}{\mu_h}, 0, 0, 0, 0, 0,\ \ 0, 0, 0, \frac{\pi_v}{\mu_v}, 0) \quad (10)$$

The human population is entirely susceptible, both environmental reservoirs are free of pathogens.

2.7 Existence of Endemic Equilibrium Point (EEP) for Co-infection Model

We examine the existence of the endemic equilibrium point (EEP) of the model (1) and this describes that the disease is currently prevalent in the population.[6]

$$E_{mT}^* = S_h^*(t), I_t^*(t), I_m^*(t), I_{mt}^*(t), R_m^*(t), R_t^*(t), R_{mt}^*(t), B_v^*(t), B_t^*(t), S_v^*(t), I_v^*(t) \quad (11)$$

$\left(\frac{ds}{dt} = 0 \text{ and } \frac{dv}{dt} = 0\right)$, gives

$$\begin{aligned}
0 &= \pi_h + \rho_t R_t + \rho_m R_m + \rho_{mt} R_{mt} - [\lambda_{t_1} + \lambda_{t_2} + \lambda_{m_1} + \lambda_{m_2}] S_h - \mu_s S \\
0 &= (\lambda_{t_1} + \lambda_{t_2}) S - (\lambda_{m_3} + \lambda_{m_4}) I_t - (\mu_h + \delta_t + a) I_t \\
0 &= (\lambda_{m_1} + \lambda_{m_2}) S - (\lambda_{t_3} + \lambda_{t_4}) I_m - (\mu_h + \delta_m + b) I_m \\
0 &= (\lambda_{t_3} + \lambda_{t_4}) I_m - (\lambda_{m_3} + \lambda_{m_4}) I_t - v((f+g+h)) I_{mt} - (\mu_h + \psi_{\delta mt}) I_{mt} \\
0 &= a I_t + v(f) I_{mt} - (\mu_h + \rho_t) R_t \\
0 &= b I_m + v(h) I_{mt} - (\mu_h + \rho_m) R_m \\
0 &= vg\, I_{mt} - (\mu_h + \rho_{mT}) R_{mt} \\
0 &= g_t B_t (1 - \frac{B_t}{K_t}) + J_t I_t + \theta_t I_{mt} - e_t B_t \\
0 &= g_v B_v (1 - \frac{B_v}{K_v}) + J_v I_v + \theta_v I_{mt} - e_v B_v \\
0 &= \pi_v + \lambda_{v_1} B_v - (e_v + \lambda_{v_2}) S_v \\
0 &= \lambda_{v_2} S_v - e_v I_v
\end{aligned} \quad (12)$$

$$\begin{aligned}
s_h^* &= \frac{\pi_h + \rho_t R_t^* + \rho_m R_m^* + \rho_{mt} R_{mt}^*}{\lambda_t + \lambda_m + \mu_s} \\
I_t^* &= \frac{(\lambda_t)(\frac{\pi_h + \rho_t R_t^* + \rho_m R_m^* + \rho_{mt} R_{mt}^*}{\lambda_t + \lambda_m + \mu_s})}{(\lambda_2 + \mu_h + \sigma_t + a)} \\
I_m^* &= \frac{(\lambda_m)(\frac{\pi_h + \rho_t R_t^* + \rho_m R_m^* + \rho_{mt} R_{mt}^*}{\lambda_t + \lambda_m + \mu_s})}{(\lambda_1 + \mu_h + \sigma_m + b)} \\
I_{mt}^* &= \frac{(\lambda_1 + \lambda_2) I_t^*}{\mu_h + \psi + V(f+g+h)}
\end{aligned} \quad (13)$$

$$R_t^* = \frac{a\, I_{t+}^*\ (vf) I_{mt}^*}{\mu_h + \rho_t}$$

$$R_m^* = \frac{b\, I_{t+}^*\ (vh) I_{mt}^*}{\mu_h + \rho_m}$$

$$R_{mt}^* = \frac{(vg) I_{mT}^*}{\mu_h + P_{mt}}$$

$$B_{t^*} = \frac{K_t(g_T - e_T) + \sqrt{(K_t^2\,(gt - e_t)^2 + 4 g_T K_T (J_t I_t^* + \theta_t I_{mt}^*)}}{2 g_t} \qquad \textbf{(13 cont.)}$$

$$B_{v^*} = \frac{K_v(g_v - e_v) + \sqrt{(K_v^2\,(g_v - e_v)^2 + 4 g_v K_v (J_v I_v^* + \theta_v I_{mt}^*)}}{2 g_v}$$

$$S_v^* = \frac{\pi_v + \lambda_{v1}\left(\frac{K_v(g_v - e_v) + \sqrt{(K_v^2\,(g_v - e_v)^2 + 4 g_v K_v (J_v I_v^* + \theta_v I_{mt}^*)}}{2 g_v}\right)}{e_v + \lambda_{v2}}$$

$$I_v^* = \frac{\lambda_{v2}\left(\frac{\pi_v + \lambda_{v1} B_v^*}{e_{Sv} + \lambda_{v2}}\right)}{e_v}$$

Where $\lambda_m = \lambda_{m_1} + \lambda_{m_2}$, $\lambda_t = \lambda_{t_1} + \lambda_{t_2}$, $\lambda_1 = \lambda_{t_3} + \lambda_{t_4}$, $\lambda_2 = \lambda_{m_3} + \lambda_{m_4}$,

$\lambda_{t_1} = \frac{\beta_{t_1}\,(I_t + \mathrm{n}_t I_{mt})}{N_h}$, $\lambda_{t_2} = \frac{\beta_{t_2}\ B_t}{B_t + K_t}$, $\lambda_{m_1} = \frac{\beta_{m_1}\,(r\, I_m + \mathrm{n}_m I_{mt})}{N_h}$, $\lambda_{m_2} = \beta_{m_2}\left(\frac{B_v}{B_v + K_v}\right) I_m$,

$\lambda_{t_3} = \frac{\beta_{t_3}\,(I_T + \mathrm{n}_t I_{mt})}{N_h}$, $\lambda_{t_4} = \frac{\beta_{t_4}\ B_t}{B_t + K_t}$, $\lambda_{m_3} = \frac{\beta_{m_3}\,(r\, I_m + \mathrm{n}_m I_{mt})}{N_h}$, $\lambda_{m_4} = \beta_{m_4}\left(\frac{B_v}{B_v + K_v}\right) I_m$

2.8 Basic Reproduction Numbers

The basic reproduction number(R_0) is a key threshold parameter in epidemiology, representing the expected number of secondary infections generated by one infectious individual in a whole susceptible population. If $R_0 < 1$, the disease tends to die out: if $R_0 > 1$, the disease may persist and become endemic. [9]

2.9 Malaria-Only Model

$$\left.\begin{aligned}
\frac{dS_h}{dt} &= \pi_h + P_m R_m - (\lambda_{m_1} + \lambda_{m_2}) S - \mu_s \\
\frac{dI_m}{dt} &= (\lambda_{m_1} + \lambda_{m_2})\, S - (\mu_h + \delta_m + b)\, I_m \\
\frac{dR_m}{dt} &= b\, I_m - (\mu_h + P_m) R_m \\
\frac{dB_v}{dt} &= g_v B_v \left(1 - \frac{B_t}{K_t}\right) + J_v I_v - e_v B_v \\
\frac{dS_v}{dt} &= \pi_v + \lambda_{v_1} B_v - (\mu_{Sv} + \lambda_{v_2})\, S_v \\
\frac{dI_v}{dt} &= \lambda_{v2} S_v - e_v I_v
\end{aligned}\right\} \qquad (14)$$

$N_h(t) = S_h(t) + I_m(t) + R_m(t)$, $N_v(t) = S_v(t) + I_v(t) + B_v(t)$

2.10 Typhoid-Only Model

$$\begin{aligned}
\frac{dS_h}{dt} &= \pi_h + \rho_t R_t - [\lambda_{t_1} + \lambda_{t_2}] S - \mu_s S \\
\frac{dI_t}{dt} &= (\lambda_{t_1} + \lambda_{t_2})\, S - (\mu_h + \delta_T + a)\, I_t \\
\frac{dR_t}{dt} &= a I_t - (\mu_h + P_t)\, R_t
\end{aligned} \qquad (15)$$

$\frac{dB_t}{dt} = g_t B_t(1 - \frac{B_t}{K_t}) + J_t I_t - e_t B_t$

With total human population:

$N_h(t) = S_h(t) + t(t) + R_v(t) + B_t(t)$

2.11 Basic Reproduction Numbers

The basic reproduction number(R_0) is a key threshold parameter in epidemiology, representing the expected number of secondary infections generated by one infectious individual in a whole susceptible population. If $R_0 < 1$, the disease tends to die out: if $R_0 > 1$, the disease may persist and become endemic.[6]

2.12 Malaria Only Reproduction Number(R_0^m)

The basic reproduction number(R_0^m) for malaria Sub- model [9].

We rearrange the model equation in (15) using the method mentioned [6]. And this is obtained by taking the spectra radius of the matrix $FV^{-1} = (\rho(FV^{-1}))$.

$$F = \begin{bmatrix} (\lambda_{m_1} + \lambda_{m_2})S_h \\ g_v B_v\left(1 - \frac{B_v}{K_v}\right) \\ 0 \end{bmatrix}, V = \begin{bmatrix} (\mu_h + \delta_t + b)I_m \\ -J_v I_v + e_v I_v \\ \lambda_{v_2} S_v + e_v I_v \end{bmatrix} \quad (16)$$

$$Fv^{-1} - \lambda I = \begin{bmatrix} \frac{r.\beta_{m_1}.\pi_h}{\mu_h N_h(\pi_h + \delta_m + b)} - \lambda & 0 & 0 \\ 0 & \frac{g_v}{e_v} - \lambda & \frac{g_v J_v}{e_v^2} \\ 0 & 0 & 0 - \lambda \end{bmatrix}$$

The eigenvalues of the jacobians(J_{E_0}) is given as

$\lambda = 0$, $\lambda = \frac{g_v}{e_v}$, $\lambda = \frac{r\beta_{m_1}\pi_h}{\mu_h N_h(\pi_h + \delta_{m+} b)}$

$$R_0^m = \frac{r\beta_{m_1}\pi_h}{\mu_h N_h(\pi_h + \delta_m + b)} \quad (17)$$

2.13 Typhoid- Only Reproduction Number(R_0^t)

The basic reproduction number denoted by(R_0^t) [9], following same method as shown in (2.12) above.

$R_0^T = \rho(FV^{-1}) = \text{Max}[R_0^T]$, with ρ the spectral radius of FV^{-1}, is been given by

$$R_0^t = \left[\left[\lambda = \frac{1}{2}\frac{1}{kGe_t u}\left[Gkug_t + G\pi_h\beta_{t_1}e_t + \sqrt{4Gku^2 J_t\pi_h\beta_{t_2}e_t^2 + G^2k^2u^2g_t^2 - 2G^2ku\pi_h\beta_{t_1}e_t g_t + G^2\pi_h^2\beta_{t_1}^2 e_t^2}\right]\right]\right] \quad (18)$$

Where:

$u = \mu_h N_h$, $k = \mu_h + \delta_t + a$, $G = k_t e_t \mu_h$

2.14 Co-infection Reproduction Number (R_0^{mt})

The basic reproduction number denoted by (R_0^{mt}) [9], following same method as shown in (2.12) above, we arrived at the following results:

$$\rho(FV^{-1}) = \text{Max}\left(\frac{1}{2}\frac{D_1 + \sqrt{D_2 - 2D_3 + D_4 + 2D_5 + D_6}}{\mu_h N_h Z_t k_t e_t}, \frac{r\beta_{m_1}\pi_h}{\mu_h N_h Z_m}\right) \quad (19)$$

$Z_t = \mu_h + \delta_t + a$, $Z_m = \mu_h + \delta_m + b$, $D_1 = N_h Z_t g_t k_t \mu_h + \beta_{t_2}\pi_h J_t N_h + \beta_{t_1}\pi_h k_t e_t$,

$D_2 = N_h^2 Z_t^2 g_t^2 k_t^2 \mu_h^2 + 2J_t N_h^2 \pi_h Z_t \beta_{t_2} g_t k_t \mu_h$, $D_3 = N_h \pi_h \beta_{t_1} e_t g_t k_t^2 N_h$, $D_4 = J_t^2 N_h^2 \pi_h^2 Z_t \beta_{t2}^2$, $D_5 = 2\left(J_t N_h \pi_h^2 \beta_{t_1} \beta_{t_2} e_t k_t\right)$, $D_6 = \pi_h^2 \beta_{t1}^2 e_t^2 k_t^2$

2.15 Sensitivity Analysis of the Model Parameters

Sensitivity analysis is commonly used to discover parameters that have a high impact on the basic reproduction number R_0. It allow us to measure the relative change in a stable variable when a parameter changes. We employ the method outlined by [10], and we make use of **Maple 18 software.** We used the Normalized Forward Sensitivity Index (NFSI) methods as adopted in [10], were the sensitivity index of R_0 with respect to the parameter *p* is denoted by:

$$S_p^{R_0} = \frac{\partial R_0}{\partial P} \times \frac{p}{R_0} \quad (20)$$

Where p is the Parameter in theR_0 .

If $S_p^{R_0} < 0$ for the parameter *p*, it means increasing the parameter *p*, decreases the basic reproduction numberR_0, but if $S_p^{R_0} > 0$ for parameter *p*, means increasing parameter *p*, increases since $R_0 = \max\{R_0^t, R_0^m\}$, we obtain the sensitivity analysis of R_0^t and R_0^m seperately, with respect to each parameter to decide the influential parameter on the environment transmission of salmonella typhi bacteria, breeding vectors / pathogens and co-infection of malaria and typhoid fever disease as follows.

$$R_0^t = \left[\left[\frac{1}{2}\frac{1}{kGe_t u}\left[Gkug_t + G\pi_h\beta_{t_1}e_t + \sqrt{4Gku^2J_t\pi_h\beta_{t_2}e_t^2 + G^2k^2u^2g_t^2 - 2G^2ku\pi_h\beta_{t_1}e_tg_t + G^2\pi_h^2\beta_{t_1}^2e_t^2}\right]\right]\right] \quad (21)$$

Where;

$u = \mu_h N_h,\ k = \mu_h + \delta_t + a,\ G = k_t e_t \mu_h\ ,\ N_h = \frac{\pi_h}{\mu_h}$

$$R_0^m = \frac{r\beta_{m_1}\pi_h}{\mu_h N_h(\mu_h+\delta_m+b)} \quad (22)$$

Based on the data given in table 4.1 and 4.2 for typhoid fever and malaria disease on direct and indirect transmission with environmental dependency, we computed the following. We generate the following sensitivity simulation as well as the Plots presented in this section, for malaria and typhoid as follows:

2.16 Numerical Simulations

Table 2.1 Sensitivity Indices of the Model Parameter for Malaria (R_0^m)

Parameters	Description	Values	Indices
π_h	Constant recruitment rate of human population	$\mathbf{-0.99999}$	Strong Negative
$\boldsymbol{b}$	Recovery rate for human with malaria	$\mathbf{0.33333}$	Moderate Positive
δ_m	Malaria induced death rate	$\mathbf{-0.0000014603}$	Negative
r	Per biting rate of mosquito to human	$\mathbf{-0.000014603}$	Negative
β_{m1}	Direct transmission rate of malaria	1	Dominant Positive

We plotted the sensitivity indices for R_0^m values against the parameter using R-programming language as shown in Figure 2.1

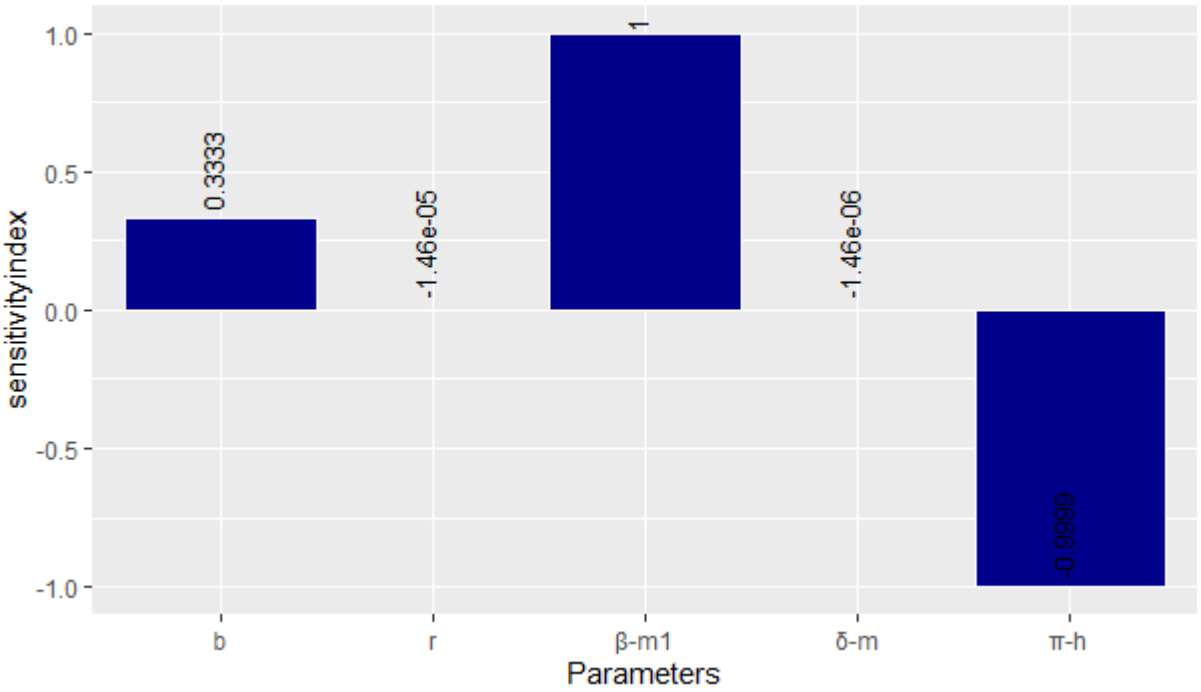

Figure 2.1: Sensitivity Plot Index of malaria

Table 2.2 Sensitivity Indices of the Model Parameter for Typhoid Fever (R_0^t)

Parameters	Description	Values	Indices
μ_h	Natural death rate for human	0.5378	Positive
J_t	Samonella typhi shedding rate into the environment	0.4601	Positive
g_t	Typhoid fever bacteria growth rate	0.0796	Small positive
e_t	Decay rate for salmonella typhi	1.4603	Positive Influential
δ_t	Typhoid fever induced death rate	0.0064	Small Positive
β_{t_2}	Indirect transmission rate for typhoid	0.4601	Positive
β_{t_1}	Direct transmission rate for typhoid	0.00052	Small Positive
π_h	Constant recruitment rate of human Population	2.9202	Most important positive
a	Recovery rate for human with typhoid	-0.4214	Moderate Negative

We plotted the sensitivity indices for R_0^t values against the parameter using R-programming language as shown in Figure 2.2

2.2: Sensitivity Plot Index of typhoid fever

Key Results and Interpretation for Sensitivity index

The sensitivity results and set of parameters that were employed, are given in table 2.1 and 2.2.

Table 2.1. For malaria sub-model shows, the induced death (δ_t) and biting rate (r) have negative influence on the basic reproduction number(R_0).

Table 2.2 of the typhoid sub-model shows that the most influential positive parameters that strongly increases disease spread are recruitment rate (π_h), thereby enhancing disease transmission. In contrast, recovery rate (a) contributes negatively and the decay rate (e_t) exhibits a positive sensitivity index because of his mathematical placement in typhoid reproduction umber (R_0), while the direct transmission pathway (β_{t_1}) has only a small effect. Biologically, faster decay should reduce disease transmission.

3.0 Optimal Control Analysis of Malaria-Typhoid Co-infection Model

The system of differential equations is numerically integrated in **Maple 17**, while the state and adjoint system were solved using the classical fourth-order Runge–Kutta (RK4) method [7-8], This time-stepping scheme is embedded within a broader Forward–Backward Sweep Method (FBSM) framework. Four time-dependent control functions were incorporated into the model to account for key intervention strategies

(a) $u_1(t)$: Treatment and hygienic control efforts targeting individuals infected with typhoid fever only;

(b) $u_2(t)$: Treatment and hygienic control efforts for individuals infected with malaria only;

(c) $u_3(t)$: Combined treatment and hygienic control efforts for individuals with malaria–typhoid co-infection;

(d) $u_4(t)$: Environmental reservoir control efforts

3.1 Optimal Control Model Equation for Malaria-Typhoid Co-Infection

The following are the optimal control model equations.

$$\left.\begin{aligned}
\frac{d_{s_h}}{dt} &= \pi_h + \rho_t R_t + \rho_m R_m + \rho_{mt} R_{mt} - \left(\lambda_{t_1} + \lambda_{t_2}\right) + \left(\lambda_{m_1} + \lambda_{m_2}\right) S_h - \mu_s \\
\frac{dI_t}{dt} &= \left(\lambda_{t_1} + \lambda_{t_2}\right) S_h - \left(\lambda_{m_3} + \lambda_{m_4}\right) I_t - \left(\mu_h + \delta_T + a + u_1\right) I_t \\
\frac{dI_m}{dt} &= \left(\lambda_{m_1} + \lambda_{m_2}\right) S_h - \left(\lambda_{t_3} + \lambda_{t_4}\right) I_m - \left(\mu_h + \delta_m + b + u_2\right) I_m \\
\frac{dI_{mt}}{dt} &= \left(\lambda_{t_3} + \lambda_{t_4}\right) I_m + \left(\lambda_{m_3} + \lambda_{m_4}\right) I_t - (v(f+g+h) + \mu_h + \psi + u_3) I_{mt} \\
\frac{dR_t}{dt} &= (a + u_1)) I_t + v(f+u_3) I_{mt} - (\mu_h + \rho_t) R_t \\
\frac{dR_m}{dt} &= (b+u_2) I_m + (v(h+u_3) I_{mt} - (\mu_h + \rho_m) R_m \\
\frac{dRm_t}{dt} &= v(g+u_3) I_{mt} - \left(\mu_h + \rho_{mT}\right) R_{mt} \\
\frac{dB_t}{dt} &= g_t B_t \left(1 - \frac{B_t}{K_t}\right) + (1-u_4)(J_t I_t + \theta_t I_{mt}) - e_t B_t \\
\frac{dB_v}{dt} &= g_v B_v \left(1 - \frac{B_v}{K_v}\right) + (1-u_4)(J_v I_v + \theta_v I_{mt}) - e_v B_v \\
\frac{dS_v}{dt} &= \pi_v + \lambda_{v_1} B_v - (e_v + \lambda_{v_2}) S_v \\
\frac{dI_v}{dt} &= \lambda_{v_2} S_v - (e_v) I_v
\end{aligned}\right\} \quad (23)$$

$$\left.\begin{aligned}
&S_h(0) \geq 0,\ I_t(0) \geq 0,\ I_m(0) \geq 0,\ I_{mt}(0) \geq 0,\ R_t(0) \geq 0,\ R_m(0) \geq 0, \\
&R_{mt}(0) \geq 0,\ B_t(0) \geq 0,\ B_v(0) \geq 0,\ S_v(0) \geq 0,\ I_v(0) \geq 0
\end{aligned}\right\} \quad (24)$$

For $0 \leq u_i \leq 1$

3.2 Formulation of Optimal Control Model for Malaria-Typhoid Co-Infection

Our aims are to reduce the number of infected individuals, the breeding vectors (or pathogens) and the salmonella typhi bacteria in the environment, while minimizing both environmental contamination costs and the expenses of implementing control strategies over the time interval $[t_0, t_f]$.

$$\text{Min J}\,(u_1, u_2, u_3, u_4) \int_{t_0}^{t_f} \left[C_1 I_t + C_2 I_m + C_3 I_{mt} + C_4 B_t + C_5 B_v + \frac{1}{2}\sum_{i=1}^{4} w_i {u_i}^2\right] dt \quad (25)$$

$$\text{Min J}\,(u_1, u_2, u_3, u_4) \int_{t_0}^{t_f} \left[C_1 I_t + C_2 I_m + C_3 I_{mt} + C_4 B_t + C_5 B_v + \frac{1}{2} w_1 (u_1)^2 + \frac{1}{2} w_2 (u_2)^2 + \frac{1}{2} w_3 (u_3)^2 + \frac{1}{2} w_4 (u_4)^2\right] dt$$

The expression $\frac{1}{2} w_1 (u_1)^2 + \frac{1}{2} w_2 (u_2)^2 \pm \frac{1}{2} w_3 (u_3)^2 + \frac{1}{2} w_4 (u_4)^2$ gives:

$$\mathcal{H}(t, x(t), u(t), \lambda_i) = \left(C_1 I_t + C_2 I_m + C_3 I_{mt} + C_4 B_t + C_5 B_v + \sum_{i=1}^{4} \frac{w_i (u_i)^2}{2}\right) + \sum_{i=1}^{11} \lambda_i g_i (t, x(t), u(t),)$$

$$= C_1 I_t + C_2 I_m + C_3 I_{mt} + C_4 B_t + C_5 B_v + \frac{w_1 (u_1)^2}{2} + \frac{w_2 (u_2)^2}{2} + \frac{w_3 (u_3)^2}{2} + \frac{w_4 (u_3)^2}{2} + \lambda_1 (\pi_h + \rho_t R_t + \rho_m R_m + \rho_{mt} R_{mt} - \left(\lambda_{t_1} + \lambda_{t_2} + \lambda_{m_1} + \lambda_{m_2}\right) S_h - \mu_s S_h) + \lambda_2 \left(\left(\lambda_{t_1} + \lambda_{t_2}\right) S_h - (\lambda_{m_3} + \lambda_{m_4}) I_t - (\delta_t + a + u_1) I_t\right) + \lambda_3 ((\lambda_{m_1} + \lambda_{m_2}) S_h - (\lambda_{t_3} + \lambda_{t_4}) I_m) - (\mu_h + \delta_m + b + u_2) I_m)) + \lambda_4 ((\lambda_{t_3} + \lambda_{t_4}) I_m + (\lambda_{m_3} + \lambda_{m_4}) I_t - (v(f+g+h) + \mu_h + \psi + u_3) I_{mt}) + \lambda_5 ((a+u_1) I_t +$$

$v(f+u_3)I_{mt}) - (\mu_h + \rho_t)R_t + \lambda_6((b+u_2)I_m + v(h+u_3)I_{mt} - (\mu_h + \rho_m)R_m) + \lambda_7(v(g+u_3)I_{mt} - (\mu_h + \rho_{mt})R_{mt}) + \lambda_8\left(g_t B_t(1-\frac{B_t}{k_t}\right) + (1-u_4)(J_t I_t + \theta_t I_{mt}) - e_t B_t) + \lambda_9\left(g_v B_v(1-\frac{B_v}{k_v}\right) + (1-u_5)(J_v I_v + \theta_v I_{mt}) - e_v B_v) + \lambda_{10}(\pi_v + \lambda_{v_1} B_v - e_v + \lambda_{v_2})S_v) + \lambda_{11}(\lambda_{v_2} S_v - e_v I_v)$ [26]

where λ_i=*1, 2… 11* are the adjoints variables

$\frac{d\lambda_i}{dt}$, *i=1, 2,…, 11.* We employ the Pontryagin's maximum principle. $\frac{d\lambda_i}{dt} = -\frac{\partial H}{\partial x_i}$

Theorem 3:

$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial S_h}$, $\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial I_t}$, $\frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial I_m}$, $\frac{d\lambda_4}{dt} = -\frac{\partial H}{\partial I_{mt}}$, $\frac{d\lambda_5}{dt} = -\frac{\partial H}{\partial R_t}$, $\frac{d\lambda_6}{dt} = -\frac{\partial H}{\partial R_m}$,

$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial R_{mt}}$, $\frac{d\lambda_8}{dt} = -\frac{\partial H}{\partial B_t}$, $\frac{d\lambda_9}{dt} = -\frac{\partial H}{\partial B_v}$, $\frac{d\lambda_{10}}{dt} = -\frac{\partial H}{\partial S_v}$, $\frac{d\lambda_{11}}{dt} = -\frac{\partial H}{\partial I_v}$

Proof:

$\frac{d\lambda_1}{dt} = \lambda_1(\lambda_{t_1} + \lambda_{t_2} + \lambda_{m_1} + \lambda_{m_2} + \mu_h) - \lambda_2(\lambda_{t_1} + \lambda_{t_2}) - \lambda_3(\lambda_{m_1} + \lambda_{m_2})$

$\frac{d\lambda_2}{dt} = -C_1 + \lambda_2(u_1 + \mu_h + \delta_t + a) - \lambda_4(\lambda_{m_3} + \lambda_{m_4}) - \lambda_5(u_1 + a) - \lambda_8((1-u_4)J_t)$

$\frac{d\lambda_3}{dt} = -C_2 + \lambda_3(u_2 + \mu_h + \delta_m + b) - \lambda_4(\lambda_{t_3} + \lambda_{t_4}) - \lambda_6(b+u_2) - \lambda_9((1-u_4)J_v) + -\lambda_{10}(e_v + \lambda_{v_2})S_v$

$\frac{d\lambda_4}{dt} = -C_3 + \lambda_4(u_3 + \mu_h + v(f+g+h)) - \lambda_7(v(g) + u_3) - \lambda_8((1-\mathrm{u}_4)J_t) - \lambda_9((1-\mathrm{u}_4)\theta_t - \lambda_{10}(e_v + \lambda_v)S_v)$

$\frac{d\lambda_5}{dt} = \lambda_5(u_h + \mu_h) - \lambda_1(p_t)$

$\frac{d\lambda_6}{dt} = \lambda_6(u_h + \mu_h) - \lambda_1(p_m)$

$\frac{d\lambda_7}{dt} = \lambda_7(u_h + \psi) - \lambda_1(p_{mt})$

$\frac{d\lambda_8}{dt} = -C_4 + \lambda_8\left(g_t(1 - \frac{2B_t}{K_t}) - e_t\right) - (\lambda_1 + \lambda_2 + \lambda_4)S_h(1-\mathrm{u}_4)\frac{\beta_{t_2} k_t}{(B_t + K_t)^2}$

$\frac{d\lambda_9}{dt} = -C_5 + \lambda_9\left(g_v\left(1 - \frac{2B_v}{K_v}\right) - e_v\right) - (\lambda_1 + \lambda_3 + \lambda_4) - S(1-\mathrm{u}_4)\frac{\beta_{m_2} B_v}{(B_v + K_v)^2}$

$\frac{d\lambda_{10}}{dt} = \lambda_{10}(e_v + \lambda_{v_2}) - \lambda_{11}\lambda_{v_2}$

$\frac{d\lambda_{11}}{dt} = \lambda_{11}(e_v) - \lambda_{10}(1-\mathrm{u}_4)(e_v + \lambda_{v_2})s_V$ (27)

and

$$\left.\begin{aligned} \frac{\partial H}{u_1^*} &= W_1 u_1 + (\lambda_2 - \lambda_5)I_t \\ \frac{\partial H}{u_2^*} &= W_2 u_2 + (\lambda_3 - \lambda_6)I_m \\ \frac{\partial H}{u_3^*} &= W_3 u_3 + (\acute{\lambda}_4 - af\acute{\lambda}_5 + bh\acute{\lambda}_6 + g\acute{\lambda}_7))I_{mt} \\ \frac{\partial H}{u_4^*} &= W_4 u_4 + \left(\lambda_8(J_t I_t + \theta_t I_{mt}) + \lambda_9(J_v I_v + \theta_v I_{mt}) - S_h\left[\left((\lambda_2 - \lambda_1)\lambda_{t_2} + (\lambda_3 - \lambda_1)\lambda_{m_2}\right)\right]\right)I_t \end{aligned}\right\} \quad (28)$$

$\frac{\partial \mathcal{H}}{\partial u_i^*} = 0$, where $u_i^* = u_1^*, u_2^*, u_3^*, u_4^*$

We obtain the following equations

$$\left.\begin{aligned}\frac{\partial \mathcal{H}}{\partial u_1^*}&=0\\ \frac{\partial \mathcal{H}}{\partial u_2^*}&=0\\ \frac{\partial \mathcal{H}}{\partial u_3^*}&=0\\ \frac{\partial \mathcal{H}}{\partial u_4^*}&=0\end{aligned}\right\} \tag{29}$$

Solving for u_1, u_2, u_3, u_4, we have:

$$\left.\begin{aligned}u_1&=-\frac{(\acute{\lambda}_5-\acute{\lambda}_2)I_t}{W_1}=\frac{(\acute{\lambda}_2-\acute{\lambda}_5)I_t}{W_1}\\ u_2&=-\frac{(\acute{\lambda}_6-\acute{\lambda}_3)I_m}{W_2}=\frac{(\acute{\lambda}_3-\acute{\lambda}_6)I_m}{W_2}\\ u_3&=-\frac{(af\acute{\lambda}_5+bh\acute{\lambda}_6+g\acute{\lambda}_7-\acute{\lambda}_4)I_{mt}}{W_3}=\frac{(\acute{\lambda}_4-af\acute{\lambda}_5+bh\acute{\lambda}_6+g\acute{\lambda}_7))I_{mt}}{W_3}\\ u_4&=\frac{\left(\acute{\lambda}_8(J_tI_t+\theta_tI_{mt})+\acute{\lambda}_9(J_vI_v+\theta_vI_{mt})-S_h\left[\left((\acute{\lambda}_2-\acute{\lambda}_1)\lambda_{t_2}+(\acute{\lambda}_3-\acute{\lambda}_1)\lambda_{m_2}\right)\right]\right)}{W_4}\end{aligned}\right\} \tag{30}$$

$\acute{\lambda}_i(\mathrm{t_f})=0$, where $i=1,\ldots,11$. For

$\acute{\lambda}_1(\mathrm{t_f})=\lambda_2(\mathrm{t_f})=\acute{\lambda}_3(\mathrm{t_f})=\acute{\lambda}_4(\mathrm{t_f})=\acute{\lambda}_5(\mathrm{t_f})=\acute{\lambda}_6(\mathrm{t_f})=\acute{\lambda}_7(\mathrm{t_f})=\acute{\ell}_8(\mathrm{t_f})=\acute{\lambda}_9(\mathrm{t_f})=\acute{\lambda}_{10}(\mathrm{t_f})=\acute{\lambda}_{11}(\mathrm{t_f})=0$

$\frac{\partial H}{\partial u_i}=0, i=1,2,3$ and 4 respectively, subject to the constraints

$U=\{(u_1,u_2,u_3,u_4):\ 0\le u_1\le u_{1max}, 0\le u_2\le u_{2max}, 0\le u_3\le u_{3max}$ and $0\le u_4\le u_{4max}\}$

$$u_i^*(t)=\min\left(1,max\left(0,\frac{\text{adjiont difference}}{W_i}\right)\right),\qquad i=1,2,3,4.$$

This imply control remain within [0, 1]

$$\left.\begin{aligned}u_1&=min\left(1,max\left(0,\frac{(\acute{\lambda}_2-\acute{\lambda}_5)I_t}{W_1}\right)\right)\\ u_2&=min\left(1,max\left(0,\frac{(\acute{\lambda}_3-\acute{\lambda}_6)I_m}{W_2}\right)\right)\\ u_3&=min\left(1,max\left(0,\frac{(\acute{\lambda}_4-af\acute{\lambda}_5+bh\acute{\lambda}_6+g\acute{\lambda}_7)I_{mt}}{W_3}\right)\right)\\ u_4&=min\left(1,\max\left(\left(0,\frac{\acute{\lambda}_8(J_tI_t+\theta_tI_{mt})+\acute{\lambda}_9(J_vI_v+\theta_vI_{mt})-S_h[(\lambda_{t_2}(\acute{\lambda}_1-\acute{\lambda}_2)+(\acute{\lambda}_1-\acute{\lambda}_3)\lambda_{m_2})]}{W_4}\right)\right)\right)\end{aligned}\right\} \tag{31}$$

Table 2. Estimated values of the parameters used in Optimal Control Numerical Experiments

Variables	Values	Source	Parameters	Values	Source
$S_h(t)$	200000	Assumed	e_t	0.1	Assumed
$S_v(t)$	1800	Assumed	e_v	0.1	Aria-Castro(2020)
$I_t(t)$	1000	Assumsed	θ_t	0.02	Assumed
$I_m(t)$	20000	Assumed	θ_v	0.02	Assumed
$I_{mt}(t)$	9000	Assumed	π_h	20544	World Bank (2022) and WHO(2023)
$I_v(t)$	500	Assumed	π_v	41000	WHO (2010-2023)
$R_1(t)$	0	Assumed	k_t	0.006	Okuonghae & Inyama (2018)

$R_2(t)$	0	Assumed	k_v	0.0065	Okuonghae & Inyama (2011)
$R_3(t)$	0	Assumed	A	0.05	LHT (2025)
$B_t(t)$	1000	Assumed	B	0.05	WHO(2023)
$B_v(t)$	1000	Assumed	F	0.1	Assumed
Parameters	**Values**	**Source**	G	0.08	Junehyul, *et al* (2017)
h	0.1	Assumed	μ_v	0.006day^{-1}	Assumed
ρ_t	0.001	Nyerere et al. (2024)	μ_h	0.00004 day^{-1}	WHO(2023)
ρ_m	0.001	Richard et al. (2024)	g_t	0.20	Assumed
ρ_{mt}	0.001	Assumed	J_t	0.05	Assumed
g_v	0.05	Aria-Castro(2020)	J_v	0.05	Assumed
k_t	10000	Assumed	β_{t_1}	0.3	Lawal et al. (2024)
k_v	10000	Aria-Castro(2020)	β_{t_2}	0.2	Lawal et al. (2024)
β_{m_3}	0.2	Akinyi et al., 2020	β_{t_3}	0.25	Lawal et al. (2024)
β_{m_4}	0.15	Akinyi et al., 2020	β_{t_4}	0.15	Lawal et al. (2024)
c_1	1	Assumed	β_{m_1}	0.25	Akinyi et al., 2020
c_2	1	Assumed	β_{m_2}	0.2	Akinyi et al., 2020
c_3	1	Assumed	w_1	50	Assumed
c_4	1	Assumed	w_2	50	Assumed
w_4	40	Assumed	w_3	75	Assumed

4.0 Numerical Solution of the Optimal Control Model [11- 14]

The control variables u_1 , u_2, u_3, and u_4 are assumed to be bounded between 0 and 1, reflecting feasible levels of intervention intensity.

The numerical implementation is carried out using Maple 17, employing the classical fourth-order Runge–Kutta (RK4) method within a forward–backward sweep framework to solve the optimal control problem. This approach, which has been successfully applied in related studies [8].

4.1 Graphical Presentation of Numerical Experiment of the Optimal Control Model

Experiment 1: Effect of control on environmental reservoir of Salmoella typhi (B_t) over time, with all controls ($u_1 \neq 0, u_2 \neq 0, u_3 \neq 0, u_4 \neq 0$), all no controls ($u_1 = u_2 = u_3 = u_4 = 0$), typhoid control and co-infection ($u_1 \neq 0, u_3 \neq 0, u_2 = u_4 = 0$), co-infection and environmental control ($u_3 \neq 0, u_4 \neq 0, u_1 = u_2 = 0$), environmental control only ($u_4 \neq 0, u_1 = u_2 = u_3 = 0$)

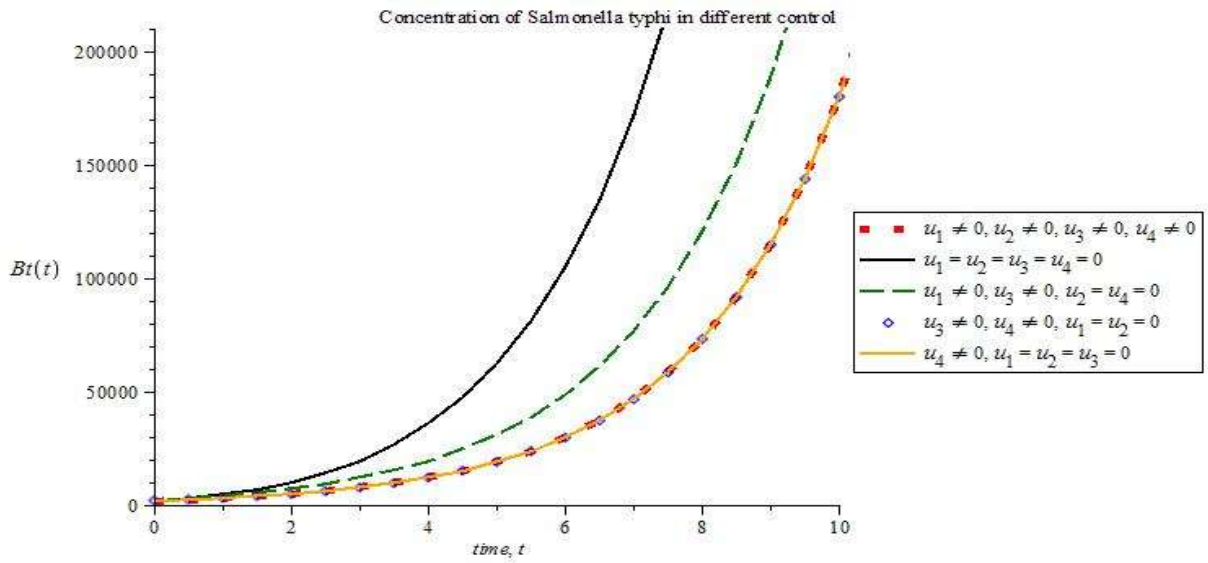

Figure 3: Prevalence of the behaviour of the salmonella typhi (B_t) population

Interpretations of Result for experiment 1:

It shows that without control ($u_1 \neq 0, u_2 \neq 0, u_3 \neq 0, u_4 \neq 0$), salmonella typhi in the environment grow explosively. All control ($u_1 \neq 0, u_2 \neq 0, u_3 \neq 0, u_4 \neq 0$) and only environmental control ($u_4 \neq 0,\ u_1 = u_2 = u_3 = 0$) Strategies keep the concentration much lower, proving that environmental intervention and treatments works well.

Experiment 2: Effect of controls on environmental reservoir of breeding vectors/pathogens (B_v) over time with all controls ($u_1 \neq 0, u_2 \neq 0, u_3 \neq 0, u_4 \neq 0$), all no controls ($u_1 = u_2 = u_3 = u_4 = 0$), typhoid control and co-infection ($u_1 \neq 0,\ u_3 \neq 0,\ u_2 = u_4 = 0$), co-infection and environmental control ($u_3 \neq 0, u_4 \neq 0, u_1 = u_2 = 0$), environmental control only ($u_4 \neq 0, u_1 = u_2 = u_3 = 0$)

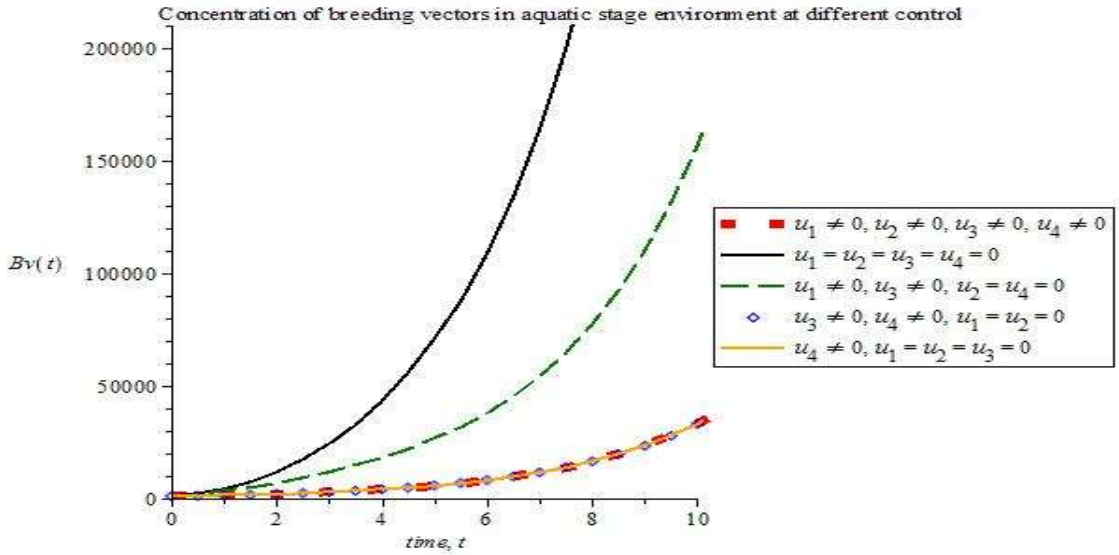

Figure 4. Prevalence of the behavior of breeding vectors/pathogens (B_v) population

Interpretations of Result for experiment 2:

Breeding vectors increases rapidly with no control ($u_1 \neq 0, u_2 \neq 0, u_3 \neq 0, u_4 \neq 0$).
Environmental control ($u_4 \neq 0, u_1 = u_2 = u_3 = 0$) alone is very effective at suppressing their growth and preventing explosion.

Experiment 3: Effect of controls of humans infected with typhoid fever (I_t) over time. with all control ($u_1 \neq 0, u_2 \neq 0, u_3 \neq 0, u_4 \neq 0$), all no controls ($u_1 = u_2 = u_3 = u_4 = 0$), malaria and typhoid control ($u_1 \neq 0, u_3 \neq 0, u_2 = u_4 = 0$), co-infection and environmental control ($u_3 \neq 0$, $u_4 \neq 0, u_1 = u_2 = 0$), environmental control only ($u_4 \neq 0, u_1 = u_2 = u_3 = 0$)

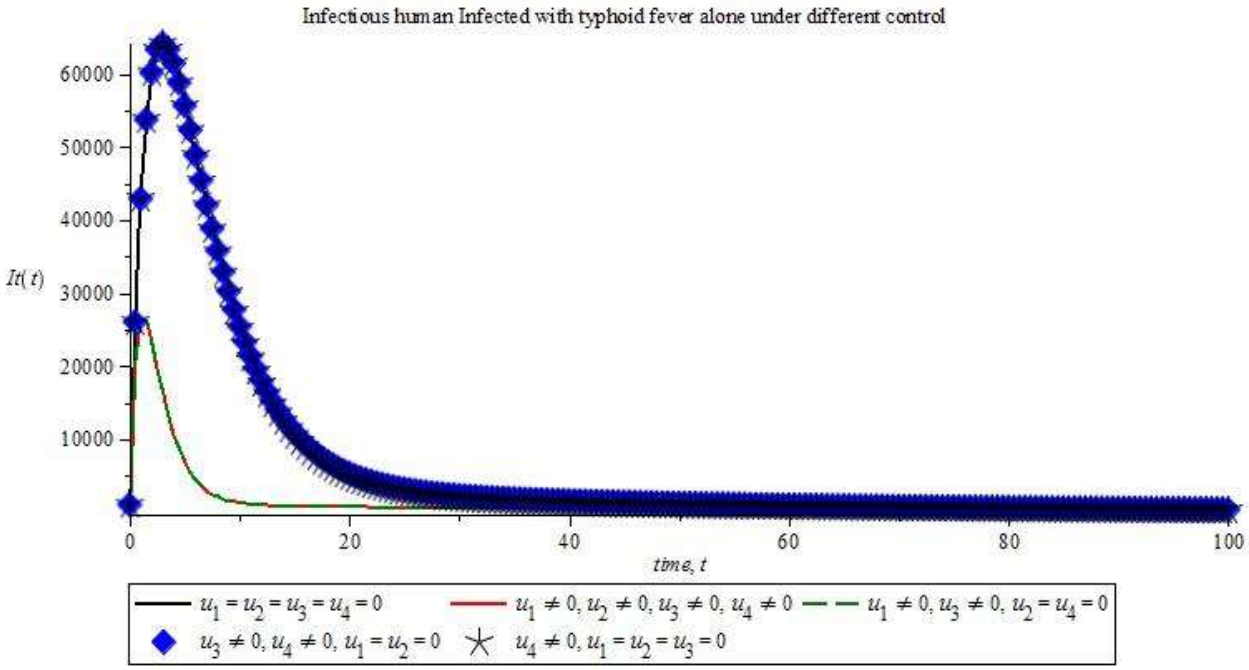

Figure 5: Prevalence of the infectious humans with only typhoid(I_t) population

Interpretations of Result for experiment 3:

Typhoid-only infection drop sharply when all controls ($u_1 \neq 0, u_2 \neq 0, u_3 \neq 0, u_4 \neq 0$)are applied. The combination of malaria-typhoid co-infection ($u_1 \neq 0, u_3 \neq 0, u_2 = u_4 = 0$) treatment plus environment control gives the best reduction ($u_4 \neq 0, u_1 = u_2 = u_3 = 0$).

Experiment 4: Effect of controls of humans infected with malaria (I_m) over time, with all controls ($u_1 \neq 0, u_2 \neq 0, u_3 \neq 0, u_4 \neq 0$), all no controls ($u_1 = u_2 = u_3 = u_4 = 0$), malaria and typhoid control ($u_1 \neq 0, u_3 \neq 0, u_2 = u_4 = 0$), co-infection and environmental control ($u_3 \neq 0, u_4 \neq 0$, $u_1 = u_2 = 0$), environmental control only ($u_4 \neq 0, u_1 = u_2 = u_3 = 0$)

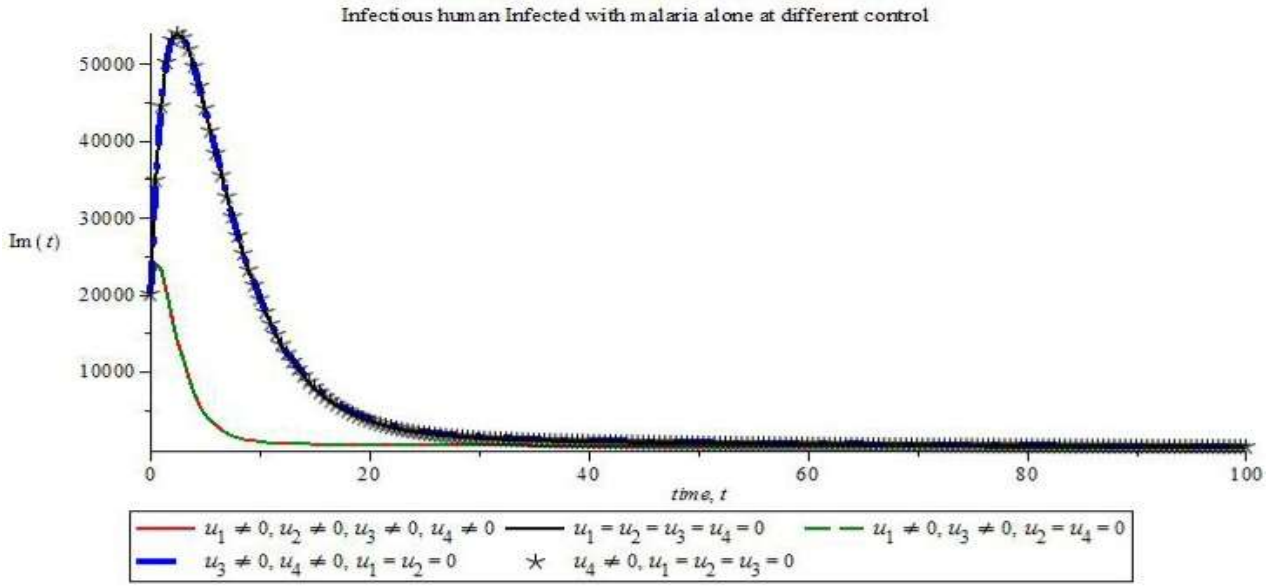

Figure 6: Prevalence of the infectious humans with malaria only (I_m) population

Interpretations of Result for experiment 4:

Malaria-only infection drop sharply when all controls ($u_1 \neq 0, u_2 \neq 0, u_3 \neq 0, u_4 \neq 0$)are applied. The combination of malaria-typhoid co-infection ($u_1 \neq 0$, $u_3 \neq 0$, $u_2 = u_4 = 0$) treatment plus environment control shows the strongest impart in reducing infected humans. ($u_4 \neq 0$, $u_1 = u_2 = u_3 = 0$).

Experiment 5: Effect of co-infection infected individual (I_{mt}) with malaria ad typhoid fever over time, with control ($u_1 \neq 0, u_2 \neq 0, u_3 \neq 0, u_4 \neq 0$), all no controls ($u_1 = u_2 = u_3 = u_4 = 0$), malaria and typhoid control ($u_1 \neq 0$, $u_3 \neq 0$, $u_2 = u_4 = 0$), co-infection and environmental control ($u_3 \neq 0$, $u_4 \neq 0$, $u_1 = u_2 = 0$), environmental control only ($u_4 \neq 0$, $u_1 = u_2 = u_3 = 0$)

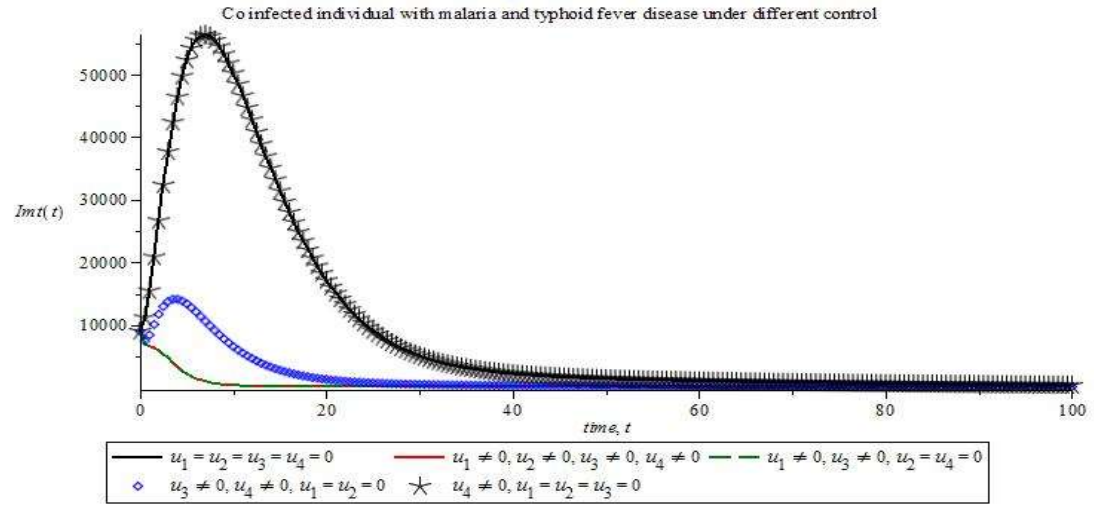

Figure 7: Prevalence of the infectious humans with co-infection (I_{mt}) population

Interpretations of Result for experiment 5:

Malaria-typhoid cases drop quickly when there is combination of full control ($u_1 \neq 0, u_2 \neq 0, u_3 \neq 0, u_4 \neq 0$), environmental control and treatment ($u_4 \neq 0$, $u_1 = u_2 = u_3 = 0$)

5.0 Discussion of Results for Optimal Control

The optimal control profiles further reveal that control intensities are highest at the early stages of intervention and gradually decline over time. This behavior confirms that optimal control theory

successfully balances disease reduction with intervention cost, thereby satisfying the cost-minimization objective embedded in the Pontryagin Maximum Principle.

5.2 Conclusion

The findings of this study demonstrate that malaria–typhoid co-infection dynamics are strongly influenced by environmental factors, particularly poor sanitation, contaminated water sources, and favorable conditions for vector breeding. The inclusion of environmental reservoirs in the model provides a more realistic representation of disease transmission compared to models that focus solely on human-to-human or vector-to-human interactions.

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