

CONVERGENCE OF ADAPTIVE TIME-VARYING MARKOV CHAINS UNDER DYNAMIC ENVIRONMENTS

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ARTICLE INFO

Article history:

Received xxxxx

Revised xxxxx

Accepted xxxxx

Available online xxxxx

Keywords:

Non-homogeneous
Markov chain,
Weak ergodicity,
Dynamic environment,
Strong ergodicity,
Dobrushin coefficient

ABSTRACT

This study develops an adaptive time-varying Markov chain model for stochastic systems operating in dynamic environments. The model combines adaptive parameter updating with non-homogeneous transition probabilities to capture learning and environmental changes. Using weak ergodicity, strong ergodicity, and the Dobrushin coefficient, convergence properties are established, demonstrating the existence of a unique stationary distribution and asymptotic stability under suitable conditions. An illustrative numerical example validates the theoretical results under a periodic environment. The proposed framework extends adaptive and non-homogeneous Markov chain theory and provides a basis for modeling evolving stochastic systems.

1. Introduction

Markov chain theory is widely used to model stochastic systems that evolve through transitions between states. Classical Markov chains satisfy the Markov property, where future transitions depend only on the current state, and assume fixed transition probabilities. However, many real-world systems, including ecological, epidemiological, communication, and financial systems, evolve in dynamic environments with changing transition probabilities. This has led to the development of adaptive and time-varying Markov chain models that better capture evolving stochastic processes [13]. Adaptive Markov chains have received considerable attention because they allow transition probabilities to evolve through learning or updating mechanisms. This makes them suitable for stochastic systems operating in changing environments. Early work by [1] established fundamental ergodic and convergence properties of adaptive Markov chains. Later, [18] introduced the concepts of diminishing adaptation and containment, providing key conditions for convergence.

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DOI: <https://doi.org/10.60787/tnamp.v25.743>

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Further studies by [1] showed that adaptive algorithms can improve convergence while preserving asymptotic stability and ergodicity. Adaptive MCMC methods have been widely studied for their theoretical properties, algorithmic design, and practical applications [12, 8]. More recently, [2] and [10] investigated their convergence rates, computational efficiency, and mixing behaviour. Another important extension is the time-varying (non-homogeneous) Markov chain, in which transition probabilities change over time to capture seasonal, environmental, and policy effects. In this area, [6] established convergence results for evolving transition structures under suitable regularity and stability conditions. Recent studies have further established convergence¹ theory for adaptive and time-varying Markov chains. Strong ergodicity in non-homogeneous chains was examined by [20], while [3] established weak convergence and weak laws of large numbers for adaptive chains. Other contributions include recurrence theory [7], ergodicity of asymptotically periodic processes [16], convergence and mixing-time bounds [5, 14], estimation methods for continuous-time non-homogeneous chains [21], and structural break detection in time-inhomogeneous Markov chains [22]. Despite these advances, most studies have considered adaptive MCMC algorithms or time-varying Markov chains separately, with limited attention to adaptive time-varying Markov chains and their strong ergodicity, weak ergodicity, and convergence properties under dynamic environments. Thus, there is a need for a unified framework that incorporates adaptive learning mechanisms with time-varying environmental influences and examines their joint impact on convergence properties. This gap is addressed in this study by developing an adaptive time-varying Markov chain model under dynamic environmental conditions and establishing conditions for weak ergodicity, strong ergodicity, and convergence.

2.0 Methodology

2.1 Research Design

This study adopts a theoretical approach to investigate the convergence of adaptive time-varying Markov chains under dynamic environments. The analysis is based on ergodic theory [19], stochastic approximation, and stochastic matrix products [17, 11], with an illustrative example validating the theoretical results.

2.2 Adaptive Time-Varying Markov Chain Model

For a discrete stochastic process

$$\{X_n\}_{n \geq 0} \quad (1)$$

defined on a finite state space

$$S = \{1, 2, \dots, m\}.$$

The process progresses according to a sequence of time-dependent transition probability matrices

$$P_n = (p_{ij}^{(n)}), \quad n = 0, 1, 2, \dots, \quad (2)$$

$$p_{ij}^{(n)} = P(X_{n+1} = j \mid X_n = i), \quad (3)$$

for $i, j \in S$.

Unlike classical homogeneous Markov chains, the proposed model allows transition probabilities to vary with adaptive parameters and environmental conditions. The transition matrix at time n is given by:

$$P_n = P(\theta_n, E_n) \quad (4)$$

P_n is the matrix of transition probability at time, n ; θ_n , vector of adaptive parameters; and E_n the state of the environment at time, n .

Let μ_0 be the chain's initial probability distribution. Then at time n , the distribution of states is defined as:

$$\mu_n = \mu_0 \prod_{k=1}^n P_k \quad (5)$$

The goal is to investigate whether the sequence, μ_n converges to a limiting distribution as $n \rightarrow \infty$, despite the environmental variation and continual adaptation.

2.3 Environmental Dynamics

The process of environmental dynamics is modeled with the sequence, $\{E_n\}_{n \geq 0}$, which may evolve stochastically or deterministically.

For a stochastic environment, the evolution of environmental conditions is described by

$$E_n = g(E_{n-1}, \varepsilon_n) \quad (6)$$

Where $g(\cdot)$ is a measurable environmental transition function; ε_n represents a sequence of independent random disturbances. This formulation allows environmental conditions to fluctuate and influence the transition probabilities of the Markov chain [13].

2.4 Adaptive Parameter Updating Mechanism

To account for learning or adaptation, the parameter vector θ_n is updated recursively using a stochastic approximation scheme:

$$\theta_{n+1} = \theta_n + \alpha_n H(\theta_n, E_n) \quad (7)$$

where α_n is the learning rate or step size; and $H(\theta_n, E_n)$ is an adaptation function describing how environmental information modifies the parameter vector.

The learning sequence satisfies the standard stochastic approximation conditions $\alpha_n > 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$.

These conditions ensure that adaptation continues indefinitely while the magnitude of parameter updates gradually decreases, thereby promoting convergence [17] and [12].

2.5 Convergence Analysis

The convergence behavior of the adaptive Markov chain is studied using the theory of non-homogeneous Markov chains and ergodic stochastic processes.

2.5.1 Weak Ergodicity

A sequence of transition matrices $\{P_n\}$ is said to be weakly ergodic if

$$\lim_{n \rightarrow \infty} \sup_{i,j} \|e_i P_1 P_2 \cdots P_n - e_j P_1 P_2 \cdots P_n\| = 0 \quad (8)$$

where e_i and e_j are unit probability vectors corresponding to states i and j .

Weak ergodicity implies that the influence of the initial state diminishes over time, so that trajectories starting from different states become increasingly similar [9].

2.5.2 Strong Ergodicity

The chain is strongly ergodic if there exists a probability distribution

$$\pi = (\pi_1, \pi_2, \dots, \pi_m) \quad (9)$$

such that

$$\lim_{n \rightarrow \infty} \|\mu_0 P_1 P_2 \cdots P_n - \pi\| = 0. \quad (10)$$

Strong ergodicity guarantees convergence to a unique limiting distribution regardless of the initial distribution [15].

2.6 Coefficient of Ergodicity

To quantify convergence and contraction of probability distributions, the Dobrushin coefficient of ergodicity is employed. For a stochastic matrix P ,

$$\tau(P) = \frac{1}{2} \max_{i,j} \sum_{k=1}^m |p_{ik} - p_{jk}|. \quad (11)$$

The coefficient satisfies $0 \leq \tau(P) \leq 1$.

A smaller value of $\tau(P)$ indicates stronger contraction of probability distributions. Weak ergodicity is established when

$$\tau(P_1 P_2 \cdots P_n) \rightarrow 0, \quad n \rightarrow \infty. \quad (12)$$

The coefficient therefore serves as an important tool for evaluating convergence rates [4].

2.7 Asymptotic Stability Analysis

The sequence of matrix products is used to examine the long-run stability of the adaptive process.

$$Q_n = P_1 P_2 \cdots P_n. \quad (13)$$

The existence of the limit

$$Q = \lim_{n \rightarrow \infty} Q_n \quad (14)$$

Under suitable assumptions, the matrix Q describes the long-term behavior of the Markov chain.

2.7.1 Matrix Contraction Rate

r_n define the rate of contraction, thus

$$r_n = \tau(P_1 P_2 \cdots P_n) \quad (15)$$

which measures the degree of convergence between trajectories originating from different initial states [5].

3.0 Results and Discussion

This section presents the theoretical results of the adaptive time-varying Markov chain model. The analysis focuses on convergence, ergodicity, and long-run stability under adaptive parameter updates and dynamic environmental conditions.

3.1 Theoretical Convergence Results

3.1.1 Assumptions

Assumption A1 (Uniform Positivity)

There exists a constant $\delta > 0$ such that

$$p_{ij}^{(n)} \geq \delta \quad (16)$$

for all states $i, j \in S$ and all $n \geq 0$.

This assumption ensures every state can eventually communicate with every other state.

Assumption A2 (Diminishing Adaptation)

The adaptive parameters satisfy

$$\theta_{n+1} = \theta_n + \alpha_n H(\theta_n, E_n) \quad (17)$$

With $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$.

This guarantees that adaptation continues while the update magnitude gradually decreases.

Assumption A3 (Bounded Environmental Variation)

There exists a constant $M > 0$ such that

$$\|E_n\| \leq M \quad (18)$$

for all n . This prevents environmental shocks from becoming arbitrarily large.

Assumption A4 (Transition Matrix Convergence)

The sequence of transition matrices satisfies

$$P_n \rightarrow P^*$$

as $n \rightarrow \infty$, where P^* is a stochastic matrix.

Proposition 3.1 (Bound on the Dobrushin Coefficient)

For every stochastic matrix P_n ,

$$\tau(P_n) = \frac{1}{2} \max_{i,j} \sum_{k=1}^m |p_{ik}^{(n)} - p_{jk}^{(n)}| \quad (19)$$

Under Assumption A1,

$$\tau(P_n) \leq 1 - \delta \quad (20)$$

Proof

Since every entry satisfies

$$p_{ik}^{(n)} \geq \delta \quad (21)$$

we obtain

$$\sum_{k=1}^m \min(p_{ik}^{(n)}, p_{jk}^{(n)}) \geq \delta \quad (22)$$

Using the alternative representation of Dobrushin's coefficient,

$$\tau(P_n) = 1 - \min_{i,j} \sum_{k=1}^m \min(p_{ik}^{(n)}, p_{jk}^{(n)}) \quad (23)$$

it follows that, $\tau(P_n) \leq 1 - \delta$. Hence the result holds

Theorem 3.1 (Weak Ergodicity)

Under Assumptions A1–A3, the adaptive time-varying Markov chain is weakly ergodic.

Proof

By Proposition 3.1,

$$\tau(P_n) \leq 1 - \delta.$$

The coefficient of ergodicity satisfies

$$\tau(P_1 P_2 \cdots P_n) \leq \prod_{k=1}^n \tau(P_k) \quad (24)$$

Therefore

$$\tau(P_1 P_2 \cdots P_n) \leq (1 - \delta)^n$$

Since

$$0 < 1 - \delta < 1, \quad (1 - \delta)^n \rightarrow 0.$$

Hence

$$\tau(P_1 P_2 \cdots P_n) \rightarrow 0 \quad (25)$$

By the characterization theorem for weak ergodicity,

$$\sup_{i,j} \|e_i P_1 P_2 \cdots P_n - e_j P_1 P_2 \cdots P_n\| \rightarrow 0 \quad (26)$$

Thus, the chain is weakly ergodic.

Proposition 3.2 (Convergence of Adaptive Parameters)

Under Assumptions A2 and A3, the adaptive parameter sequence

$$\{\theta_n\} \quad (27)$$

converges almost surely to a limit

$$\theta^* \quad (28)$$

Proof

The update equation is

$$\theta_{n+1} = \theta_n + \alpha_n H(\theta_n, E_n) \quad (29)$$

Since $\sum \alpha_n = \infty$ and $\sum \alpha_n^2 < \infty$, the Robbins–Monro stochastic approximation theorem applies. Because H is bounded through Assumption A3, the stochastic approximation sequence remains stable. Consequently, $\theta_n \rightarrow \theta^*$ almost surely.

Theorem 3.2 (Strong Ergodicity)

Suppose Assumptions A1–A4 hold. Then there exists a unique probability distribution, π such that

$$\lim_{n \rightarrow \infty} \mu_n = \pi \quad (30)$$

Proof

From Proposition 3.2,

$$\theta_n \rightarrow \theta^*$$

Since environmental fluctuations are bounded,

$$P_n = P(\theta_n, E_n) \rightarrow P^* \quad (31)$$

Because P^* is irreducible and aperiodic under Assumption A1, the Perron–Frobenius theorem guarantees a unique stationary distribution, π satisfying

$$\pi P^* = \pi \quad (32)$$

From Theorem 3.1 the chain is weakly ergodic.

Moreover, convergence of transition matrices to P^* implies the conditions for strong ergodicity of non-homogeneous Markov chains. Therefore

$$\lim_{n \rightarrow \infty} \mu_0 P_1 P_2 \cdots P_n = \pi \quad (33)$$

Hence the chain is strongly ergodic.

Theorem 3.3 (Asymptotic Stability)

Let

$$Q_n = P_1 P_2 \cdots P_n \quad (34)$$

Under Assumptions A1–A4,

$$Q_n \rightarrow Q,$$

where

$$Q = \begin{pmatrix} \pi \\ \pi \\ \vdots \\ \pi \end{pmatrix} \quad (35)$$

Proof

Strong ergodicity implies

$$e_i Q_n \rightarrow \pi$$

for every state i .

Therefore, every row of Q_n converges to the same vector π .

Hence

$$Q_n \rightarrow Q \quad (36)$$

The limiting matrix is rank one and represents the equilibrium structure of the adaptive process [23].

3.1.2 Model Assumptions and Limitations

The theoretical results rely on several assumptions that may not hold exactly in real-world applications. Uniform positivity (Assumption A1) guarantees communication between all states but may be restrictive when transitions are impossible or extremely rare. The diminishing adaptation condition (Assumption A2) ensures stable parameter updates, although practical adaptive algorithms may not always satisfy the required step-size conditions. Bounded environmental variation (Assumption A3) excludes abrupt or extreme environmental shocks, while transition matrix convergence (Assumption A4) assumes that the adaptive process eventually stabilizes. Consequently, the convergence results are primarily applicable to systems with moderate environmental variability and gradual adaptation.

3.2 Illustrative Example to Verify the Theoretical Convergence Results

To demonstrate the theoretical results developed in the previous section, consider a three-state adaptive Markov chain defined on the state space

$$S = \{1,2,3\},$$

where State 1 = Low activity, State 2 = Moderate activity and State 3 = High activity
The initial probability distribution is $\mu_0 = (1,0,0)$. Thus, the process starts entirely in State 1.

3.2.1 Environmental Dynamics and Adaptive Parameter Evolution

For the numerical example, Equation (6) is specialized to a deterministic periodic environment:

$$E_n = \sin(0.1n)$$

where n is the time index. This sinusoidal function represents periodic environmental variation and serves as a deterministic realization of the general environmental model in Section 2.3. The resulting environmental values are used to update the adaptive transition probabilities during the simulation. Let the adaptive parameter satisfy

$$\theta_{n+1} = \theta_n + \alpha_n(0.5 - E_n),$$

with

$$\alpha_n = \frac{1}{n+1}$$

and initial value

$$\theta_0 = 0.50.$$

The first few environmental values are:

Table 1: Evolution of the first few environmental Adaptive Parameter values

n	E_n	α_n	θ_n	θ_{n+1}
0	0	1	0.5	1
1	0.099833	0.5	1	1.200083
2	0.198669	0.333333	1.200083	1.300527

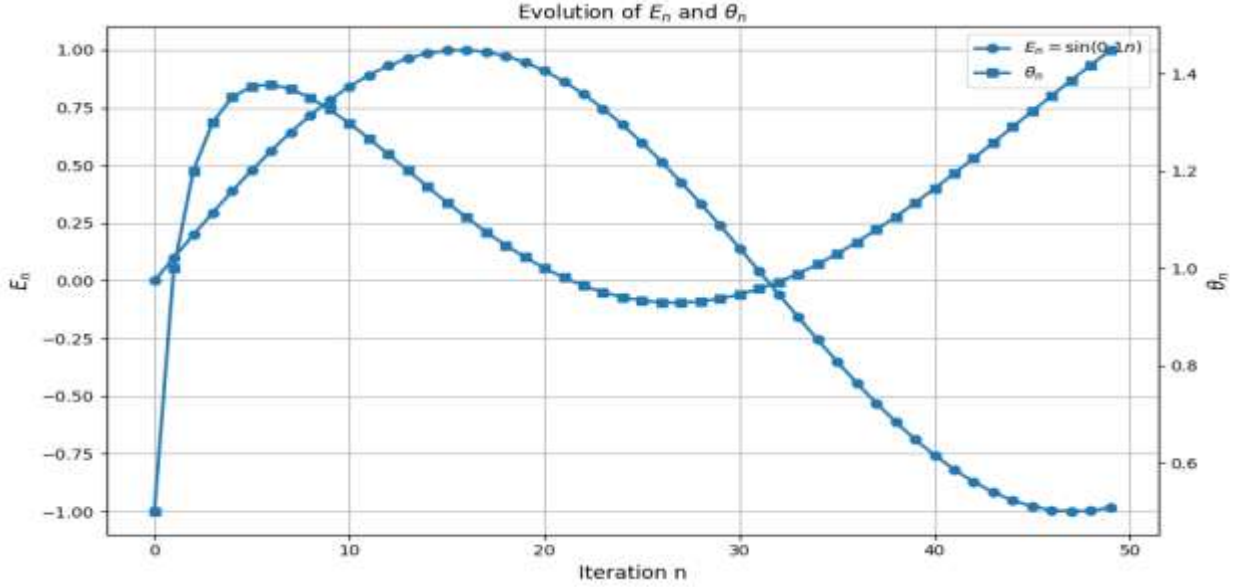


Figure 1: Environmental and Adaptive Parameter Evolution Over the Simulation Period

The evolution of θ_n and E_n , during 50 iterations, is shown in Figure 1. The smooth oscillation pattern of the sinusoidal process θ_n increases from 0 to around 1 about $n=16$, then decreases and turns negative after $n \approx 31$. Initially, θ_n increases rapidly from its starting value because $E_n < 0.5$, making the adjustment term positive. Near $n=6$, the parameter reaches a maximum of roughly 1.38. The update term turns negative as E_n keeps rising above 0.5, which causes θ_n to progressively decrease until it reaches its minimum value of roughly 0.93 around $n=26$. The figure shows that θ_n remains stable while adapting to long-term environmental changes. This supports stochastic approximation theory, demonstrating that the adaptive mechanism tracks changing conditions while maintaining stable convergence.

3.2.2 Evolution of the Transition Matrix P_n

Suppose the transition matrix is defined as

$$P_n = \begin{pmatrix} 0.6 - 0.05E_n & 0.3 + 0.03E_n & 0.1 + 0.02E_n \\ 0.2 & 0.5 & 0.3 \\ 0.1 + 0.02E_n & 0.3 + 0.03E_n & 0.6 - 0.05E_n \end{pmatrix}.$$

At $n = 0$

$$P_0 = \begin{pmatrix} 0.60 & 0.30 & 0.10 \\ 0.20 & 0.50 & 0.30 \\ 0.10 & 0.30 & 0.60 \end{pmatrix}.$$

At $n = 5$

Since

$$E_5 = 0.4794, \quad P_5 = \begin{pmatrix} 0.5760 & 0.3144 & 0.1096 \\ 0.2000 & 0.5000 & 0.3000 \\ 0.1096 & 0.3144 & 0.5760 \end{pmatrix}.$$

These matrices vary over time, reflecting environmental influence.

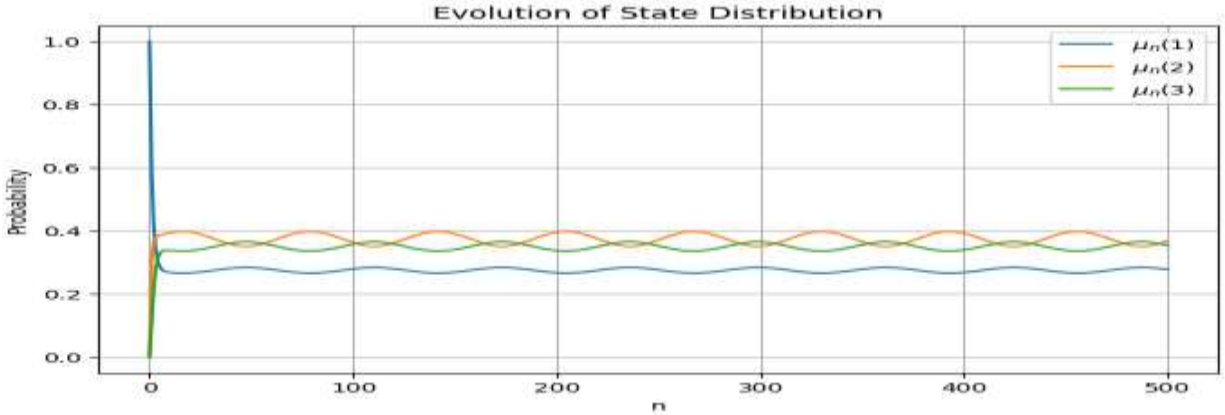


Figure 2. Evolution of the State Probability Distribution μ_n Over Successive Iterations, Showing Convergence Toward the Stationary Distribution Despite Fluctuations.

Figure 2 shows that the state distribution rapidly moves from the initial concentration in State 1 and stabilizes after a short transient period. After about 20 iterations, the probabilities oscillate within narrow bands, indicating loss of memory of the initial state. State 2 has the highest long-run probability, followed by State 3, while State 1 has the lowest. The small oscillations reflect the periodic time-varying transition structure, providing numerical evidence of ergodic behavior. This observation is consistent with the Dobrushin coefficient results, where $\tau(Q_n) \rightarrow 0$, implying weak ergodicity and loss of memory of the initial state.

Last Few Iterations

498 [0.27921837 0.36373262 0.357049]

499 [0.27841989 0.36592447 0.35565564]

500 [0.27758861 0.36819393 0.35421746]

The state vector evolves smoothly, with the first and third components decreasing while the second increases, indicating a redistribution of probability mass. The absence of oscillation or divergence suggests convergence toward a stable limiting distribution, consistent with the behavior of a time-varying stochastic system.

Table 2: Coefficient of Ergodicity

n	1	50	100
$\tau(P_n)$	0.49301	3.6082×10^{-16}	1.3878×10^{-16}

Limiting Product Matrix

At $n = 100$

$$Q_{100} \approx \begin{pmatrix} 0.27956149 & 0.36256931 & 0.35786920 \\ 0.27956149 & 0.36256931 & 0.35786920 \\ 0.27956149 & 0.36256931 & 0.35786920 \end{pmatrix}.$$

Since all rows are identical, $\tau(Q_{100}) \approx 0$, which is exactly the signature of weak ergodicity. From $\tau(P_1) = 0.4930$ to $\tau(P_{20}) \approx 10^{-7}$, the coefficient of ergodicity declines almost exponentially, and by $n=50$, it essentially reaches machine precision. As a result, the cumulative transition matrix's rows become identical, suggesting that the original state's impact vanishes. As a result, the chain is numerically weakly ergodic and shows substantial mixing behavior. For any starting

distribution $\mu_0, \mu_0 Q_n \rightarrow (0.2796, 0.3626, 0.3579)$. The limiting distribution generated by the matrix products is about $\pi = (0.2796, 0.3626, 0.3579)$.

Conclusion

This study proposes an adaptive time-varying Markov chain model that integrates adaptive learning with dynamic environmental effects. Sufficient conditions for weak ergodicity, strong ergodicity, and asymptotic stability were established using the Dobrushin coefficient and non-homogeneous Markov chain theory. The illustrative example confirmed convergence to a stable limiting distribution. The proposed framework extends classical Markov chain theory and provides a basis for modeling stochastic systems with evolving transition dynamics. Future research may consider more general adaptive mechanisms, random environments, and applications to climate, infectious disease, and ecological systems. Extending the framework to accommodate abrupt environmental changes or persistent non-stationarity remains an important direction for future research.

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